Affinely Equivalent Central Sections of Star Bodies

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Abstract

Let $K \subset \mathbb{R}^n$ be a star body about the origin and let $2 \leq k \leq n-1$. We prove, using the recently announced solution by Lu and Yang of Banach's isometric conjecture, that if all central k-dimensional sections of K are affinely equivalent, then K is an ellipsoid. This gives an affirmative answer to Problem 7.4 in Gardner's *Geometric Tomography*. The main new observation is an affine convexification lemma. In odd section dimension, a finite-order bundle obstruction can be killed after pullback by a self-map of a sphere of nonzero degree. The resulting continuous exact affine family of sections forces a fixed model section to be convex, by realizing it as a sublevel set of a supremum of affine functions. The ambient star body is then convex, so classical results of Montejano and Larman reduce the problem to the origin-symmetric linear case settled by Lu and Yang. In even section dimension, affine centroid-covariance normalization reduces the tangent-bundle structure group to the orthogonal symmetry group of a normalized model; the Euler class forces this group to act transitively, and hence all sections are ellipsoids. A two-dimensional patching argument completes the proof.

Status note. Problem 7.4 was posed by Richard J. Gardner in [2, Problem 7.4]. This draft was prepared in the course of a ChatGPT discussion initiated by Gardner. Gardner has asked not to be listed as an author and claims credit only for posing the problem. The proof below uses the August 2026 preprint of Lu and Yang [6]; consequently the conclusion depends on the correctness of that very recent result.

1 Introduction

A star body $K \subset \mathbb{R}^n$ about the origin is a compact set of the form

$$K = \{ru : u \in \mathbb{S}^{n-1},\ 0 \leq r \leq \rho_K(u)\},$$

where the radial function $\rho_K : \mathbb{S}^{n-1} \to (0, \infty)$ is continuous. No convexity or symmetry is included in this definition.

Gardner's Problem 7.4 asks whether, for $2 \leq k \leq n-1$, a star body K whose central sections

$$K \cap E, \qquad E \in \mathrm{Gr}(n,k),$$

are all affinely equivalent must be an ellipsoid; see [2, Problem 7.4] and the later problem list [3]. For convex bodies this is the geometric form of Banach's isometric subspace problem after the usual reduction to the centrally symmetric case; a long sequence of partial results culminated in the recent preprint of Lu and Yang [6], which announces a complete solution of the real Banach isometric conjecture. Montejano's survey [10] contains a detailed account of the older topological approach and of the classical results needed below. Acuaviva and Kania [1] have subsequently developed a bundle-degree method for real star bodies with Lipschitz radial function, without assuming convexity or central symmetry.

Our purpose is to prove the following statement in the full generality in which Problem 7.4 was posed.

Theorem 1.1 (Main theorem)**.** *Let $K \subset \mathbb{R}^n$ be a star body, and let $2 \leq k \leq n-1$. If the sections $K \cap E$, $E \in \mathrm{Gr}(n,k)$, are mutually affinely equivalent, then K is an ellipsoid. The ellipsoid need not be centered at the origin.*

Two elementary but useful points should be emphasized. First, an affine isomorphism between two central sections need not carry the origin to the origin; this is the essential difference between Problem 7.4 and the linear Banach problem. Second, no regularity beyond continuity of ρ_K will be assumed.

The proof is organized as follows. Section 2 constructs, for a continuous family of affinely equivalent star sections, a principal bundle of exact affine equivalences. Section 3 proves the key convexification lemma. Sections 4 and 5 settle the hyperplane problem according to the parity of the section dimension. Section 6 gives the elementary patching lemma for ellipsoidal two-dimensional sections, and Section 7 proves Theorem 1.1.

2 Affine normalization and the exact-section bundle

We first record a normalization that is valid for arbitrary star bodies. Let C be a star body in a d-dimensional Euclidean space W. Define its centroid and covariance operator by

$$b(C) = \frac{1}{\mathrm{vol}_d\, C} \int_C x\, dx, \qquad \mathrm{Cov}(C) = \frac{1}{\mathrm{vol}_d\, C} \int_C (x - b(C)) \otimes (x - b(C))\, dx.$$

Since C has nonempty interior, $\mathrm{Cov}(C)$ is positive definite. If $T(x) = Ax + c$ is an affine isomorphism, then

$$b(TC) = Ab(C) + c, \qquad \mathrm{Cov}(TC) = A\,\mathrm{Cov}(C)A^*. \tag{1}$$

Set

$$\widehat{C} = \mathrm{Cov}(C)^{-1/2}(C - b(C)). \tag{2}$$

If $C' = TC$, then

$$\widehat{C'} = Q\widehat{C}, \qquad Q = \mathrm{Cov}(C')^{-1/2} A\,\mathrm{Cov}(C)^{1/2} \in O(W, W'), \tag{3}$$

where $O(W, W')$ denotes the orthogonal isomorphisms between the two Euclidean spaces.

We shall use repeatedly the following continuity observation.

Lemma 2.1. *Let U be a topological space and let $u \mapsto C_u$ be a family of star bodies in a fixed Euclidean space W such that $(u, \theta) \mapsto \rho_{C_u}(\theta)$ is continuous. Then $u \mapsto b(C_u)$ and $u \mapsto \mathrm{Cov}(C_u)$ are continuous, and $u \mapsto \widehat{C}_u$ is continuous in the Hausdorff metric.*

Proof. On compact parameter sets, continuity of the radial functions is uniform. The polar-coordinate formulas

$$\mathrm{vol}_d(C_u) = \frac{1}{d} \int_{\mathbb{S}(W)} \rho_{C_u}(\theta)^d\, d\theta,$$

$$\int_{C_u} x\,dx = \frac{1}{d+1}\int_{\mathbb{S}(W)} \rho_{C_u}(\theta)^{d+1}\theta\,d\theta,$$

and

$$\int_{C_u} x\otimes x\,dx = \frac{1}{d+2}\int_{\mathbb{S}(W)} \rho_{C_u}(\theta)^{d+2}\theta\otimes\theta\,d\theta$$

show continuity of the centroid and covariance. The positive square root and inverse square root depend continuously on a positive-definite operator. Uniform convergence of the radial functions also gives Hausdorff convergence of C_u, and hence (2) gives the last assertion. $\square$

Fix a star body S in a Euclidean space E, and let

$$G = \mathrm{AffAut}(S) = \{g \in \mathrm{Aff}(E) : g(S) = S\}.$$

Although G is an affine group, it is compact after conjugation.

Lemma 2.2. *The affine automorphism group G of a star body S is isomorphic, as a topological group, to the compact Lie group*

$$H = \{Q \in O(E) : Q\widehat{S} = \widehat{S}\}.$$

Proof. Every $g(x) = Ax + c \in G$ fixes the centroid $b(S)$. Conjugating g by the affine normalization

$$N_S(x) = \mathrm{Cov}(S)^{-1/2}(x - b(S))$$

therefore produces a linear map. By (1), that linear map is orthogonal, and it preserves $\widehat{S}$. Conversely, conjugating an element of H by N_S^{-1} gives an affine automorphism of S. Thus G and H are conjugate. Since H is a closed subgroup of the compact Lie group $O(E)$, it is a compact Lie group. $\square$

We now construct the exact-section bundle in the affine setting. This is the continuous analogue of the exact-map bundles used in recent work on Banach's isometric conjecture; compare [1, Section 4].

Proposition 2.3 (exact affine section bundle)**.** *Let V be a Euclidean space of dimension $m+1 \geq 3$, and let $K \subset V$ be a star body whose central hyperplane sections are mutually affinely equivalent. Fix a star body $S \subset E$, $\dim E = m$, affinely equivalent to these sections. For $u \in \mathbb{S}(V)$ set*

$$\mathcal{P}_u = \{f : E \to u^\perp : f \text{ is an affine isomorphism and } f(S) = K \cap u^\perp\}.$$

Then $\mathcal{P} = \coprod_{u\in\mathbb{S}(V)} \mathcal{P}_u$ is a principal $G = \mathrm{AffAut}(S)$-bundle over $\mathbb{S}(V)$ under right composition. In particular, it has continuous exact local sections.

Proof. The right action of G on each $\mathcal{P}_u$ is free and transitive, so it remains only to construct local continuous sections.

Fix $u_0 \in \mathbb{S}(V)$. On a sufficiently small neighborhood U of u_0 choose smooth orthogonal isomorphisms

$$J_u : E \longrightarrow u^\perp.$$

Put

$$C_u = J_u^{-1}(K \cap u^\perp).$$

Then C_u is a star body about the origin and

$$\rho_{C_u}(\theta) = \rho_K(J_u\theta).$$

Consequently $u \mapsto C_u$, and by Lemma 2.1 $u \mapsto \widehat{C}_u$, are continuous.

All C_u are affinely equivalent to S, so by (3) all $\widehat{C}_u$ belong to the orbit $O(E)\widehat{S}$. Let

$$H = \{Q \in O(E) : Q\widehat{S} = \widehat{S}\}.$$

The orbit map

$$O(E)/H \longrightarrow O(E)\widehat{S}, \qquad QH \longmapsto Q\widehat{S},$$

is a homeomorphism: it is a continuous bijection from a compact space to a Hausdorff space. Hence $u \mapsto \widehat{C}_u$ determines a continuous map $U \to O(E)/H$. Since H is a closed Lie subgroup, the quotient map $O(E) \to O(E)/H$ is a locally trivial principal H-bundle (see, e.g., [12]); after shrinking U we may therefore choose continuously $Q_u \in O(E)$ with

$$Q_u\widehat{S} = \widehat{C}_u.$$

Define

$$T_u(x) = b(C_u) + \operatorname{Cov}(C_u)^{1/2} Q_u \operatorname{Cov}(S)^{-1/2}(x - b(S)).$$

Then $T_u(S) = C_u$, and

$$f_u = J_u \circ T_u : E \to u^\perp$$

is a continuous family of exact affine equivalences. Such local sections give $\mathcal{P}$ the asserted principal-bundle structure. $\square$

3 A degree lemma and affine convexification

The next lemma is the main geometric observation of the paper.

Lemma 3.1 (affine convexification). *Let V be Euclidean of dimension $N \geq 3$, let $K \subset V$ be a star body, and let $S \subset E$ be a star body with $\dim E = N - 1$. Suppose that there are continuous maps*

$$\varphi : \mathbb{S}(V) \to \mathbb{S}(V), \qquad z \longmapsto f_z \in \operatorname{Aff}(E, V),$$

where

$$f_z(y) = c_z + A_z y,$$

such that

$$f_z(E) = \varphi(z)^\perp, \qquad f_z(S) = K \cap \varphi(z)^\perp \tag{4}$$

for every z, and $\deg \varphi \neq 0$. Then S is convex. Consequently every central hyperplane section of K is convex, and K itself is convex.

Proof. Fix $y \in \partial S$. Since f_z is a homeomorphism of S onto the section in (4), $f_z(y)$ belongs to the relative boundary of $K \cap \varphi(z)^\perp$. The radial function of this section is simply the restriction of ρ_K to the unit sphere of $\varphi(z)^\perp$; hence

$$\operatorname{relbd}(K \cap \varphi(z)^\perp) = \partial K \cap \varphi(z)^\perp.$$

Thus

$$q_y(z) := f_z(y) \in \partial K \cap \varphi(z)^{\perp}.$$

Moreover $q_y(z) \neq 0$, because 0 lies in the relative interior of every central hyperplane section whereas $q_y(z)$ lies on its relative boundary. Set

$$w_y(z) = \frac{q_y(z)}{|q_y(z)|} \in \mathbb{S}(V).$$

The unit vectors $w_y(z)$ and $\varphi(z)$ are orthogonal. Therefore

$$H(z,t) = \cos\frac{\pi t}{2}\,\varphi(z) + \sin\frac{\pi t}{2}\,w_y(z), \qquad 0 \le t \le 1,$$

is a homotopy through the unit sphere from φ to w_y. Hence

$$\deg w_y = \deg \varphi \neq 0.$$

In particular, w_y is onto. Since a star body has exactly one boundary point on every positive ray from the origin, it follows that

$$\{f_z(y) : z \in \mathbb{S}(V)\} = \partial K, \qquad y \in \partial S. \tag{5}$$

Choose a nonzero linear functional $\ell \in V^*$ and put

$$h = \max_{x \in K} \ell(x).$$

Since $0 \in \operatorname{int} K$, $h > 0$. Define

$$F(y) = \max_{z \in \mathbb{S}(V)} \ell(f_z(y)) = \max_z \{\ell(c_z) + \ell(A_z y)\}, \qquad y \in E. \tag{6}$$

The function F is finite and convex, being the maximum over a compact parameter space of affine functions. By (5),

$$F(y) = h \qquad (y \in \partial S). \tag{7}$$

We claim that

$$a := F(0) = \max_z \ell(c_z) < h. \tag{8}$$

Indeed, $c_z = f_z(0)$ lies in the relative interior of $K \cap \varphi(z)^{\perp}$, because $0 \in \operatorname{int}_E S$. A point strictly inside a radial segment of a star body with positive continuous radial function is an ambient interior point; hence $c_z \in \operatorname{int} K$. The set $\{c_z : z \in \mathbb{S}(V)\}$ is compact. A nonzero linear functional cannot attain its maximum over K at an interior point, which proves (8).

Let $y \in \partial S$. If $0 \le t < 1$, convexity of F and (7)–(8) give

$$F(ty) \le (1-t)a + th < h. \tag{9}$$

If $t > 1$, then

$$y = \frac{1}{t}(ty) + \left(1 - \frac{1}{t}\right)0,$$

so convexity gives

$$h = F(y) \le \frac{1}{t}F(ty) + \left(1 - \frac{1}{t}\right)a,$$

and consequently

$$F(ty) \geq th - (t-1)a > h. \tag{10}$$

Every positive ray in E meets ∂S in exactly one point. Equations (7), (9), and (10) therefore show that

$$S = \{y \in E : F(y) \leq h\}. \tag{11}$$

Thus S is convex.

Since $\deg \varphi \neq 0$, the map φ is onto. Hence every central hyperplane section of K is an affine image of the convex body S and is convex. Finally, given $x_1, x_2 \in K$, their linear span has dimension at most two and is contained in some hyperplane through 0 because $N \geq 3$. The corresponding section of K is convex, and therefore $[x_1, x_2] \subset K$. Hence K is convex. $\square$

Remark 3.2. The translation terms c_z are essential. In the linear case one can express the gauge of S as a supremum of linear functions. Formula (11) is the affine replacement: it expresses the body itself as a sublevel set of a supremum of affine functions.

4 Odd-dimensional hyperplane sections

We need one standard topological fact. Its proof is included to make clear where parity enters.

Lemma 4.1 (killing a bundle over an odd sphere)**.** *Let $m \geq 3$ be odd and let G be a compact Lie group. If $P \to \mathbb{S}^m$ is a principal G-bundle, then there is a self-map $\varphi : \mathbb{S}^m \to \mathbb{S}^m$ of positive degree such that $\varphi^* P$ is trivial.*

Proof. Let G° be the identity component. Since G/G° is finite, $P/G^\circ \to \mathbb{S}^m$ is a finite covering. The sphere $\mathbb{S}^m$ is simply connected, so a connected component of this covering maps homeomorphically onto $\mathbb{S}^m$. Its inverse image in P is a principal G°-subbundle. Thus it suffices to treat the case in which G is connected.

A principal G-bundle over $\mathbb{S}^m$ is classified by an element

$$\alpha \in \pi_m(BG) \cong \pi_{m-1}(G).$$

Here $m - 1 > 0$ is even. For a compact connected Lie group, every positive even homotopy group is finite: after passage to a finite covering, the group is a product of a torus and a compact simply connected semisimple Lie group; the torus has no higher homotopy, while Serre's finiteness theorem applies to the even homotopy groups of the semisimple factor; see [11]. Hence α has finite order, say dividing $d > 0$.

Choose a map $\varphi : \mathbb{S}^m \to \mathbb{S}^m$ of degree d. Precomposition by a degree-d self-map acts as multiplication by d on $\pi_m(BG)$, so the class of $\varphi^* P$ is $d\alpha = 0$. Thus the pullback bundle is trivial. $\square$

The following result is the odd-parity hyperplane case.

Theorem 4.2. *Let $m \geq 3$ be odd, and let $K \subset \mathbb{R}^{m+1}$ be a star body whose central hyperplane sections are mutually affinely equivalent. Assuming the theorem of Lu and Yang [6, Theorem 1.2], K is an ellipsoid.*

Proof. Let $S \subset E$, $\dim E = m$, be a fixed model section and let $\mathcal{P} \to \mathbb{S}^m$ be the principal $G = \mathrm{AffAut}(S)$-bundle from Proposition 2.3. By Lemma 2.2, G is a compact Lie group. Lemma 4.1 gives a self-map $\varphi : \mathbb{S}^m \to \mathbb{S}^m$ of positive degree for which $\varphi^*\mathcal{P}$ is trivial. A continuous global section of the pullback is precisely a continuous family of affine maps

$$f_z : E \to \varphi(z)^\perp, \qquad f_z(S) = K \cap \varphi(z)^\perp.$$

Lemma 3.1 now shows that S and K are convex.

We may therefore use two classical results. Montejano proved that if all central hyperplane sections of a convex body are affinely equivalent, then every such section is centrally symmetric; see [10, Theorem 3.2], based on [9]. Larman's False Center Theorem then implies that either K is an ellipsoid or K is centrally symmetric about the origin; see [5] and [10, Theorem 3.3].

In the first case there is nothing to prove. In the second case every central hyperplane section $K \cap u^\perp$ is centrally symmetric about 0. Its center is unique, so any affine equivalence between two such sections must carry 0 to 0 and is therefore linear. Lu and Yang's theorem for origin-symmetric convex bodies with linearly equivalent central sections [6, Theorem 1.2] now implies that K is an ellipsoid. $\square$

5 Even-dimensional hyperplane sections

For even m the bundle class in Lemma 4.1 need not be torsion. Instead the Euler class of the tangent bundle supplies an obstruction in the opposite direction. This is the classical Gromov–Mani mechanism; compare [4, 7, 10].

Lemma 5.1. *Let $m \geq 2$ be even, let $H \subset O(m)$ be a compact subgroup, and suppose the orthonormal frame bundle of $T\mathbb{S}^m$ admits a reduction of structure group to H. Then the identity-orientation part of H acts transitively on $\mathbb{S}^{m-1}$.*

Proof. Because $\mathbb{S}^m$ is simply connected, the reduction can be refined to a reduction to an orientation-preserving subgroup $H^+ \subset SO(m)$. More explicitly, if H contains orientation-reversing elements then the quotient by $H \cap SO(m)$ is a double covering of $\mathbb{S}^m$, hence trivial; choosing a component yields an $H \cap SO(m)$-reduction. If H has no orientation-reversing elements, then $H \subset SO(m)$; after reversing the chosen orientation of the model space if necessary, the reduction lies in the oriented frame bundle.

Let

$$\chi : \mathbb{S}^{m-1} \longrightarrow SO(m)$$

be a clutching map for $T\mathbb{S}^m$. Reduction of structure group means that, up to homotopy, χ factors through H^+. Fix $a \in \mathbb{S}^{m-1}$ and let

$$e_a : SO(m) \to \mathbb{S}^{m-1}, \qquad e_a(Q) = Qa.$$

For an oriented rank-m vector bundle over $\mathbb{S}^m$, the degree of the clutching map followed by e_a is its Euler number. For $T\mathbb{S}^m$ this Euler number is

$$\langle e(T\mathbb{S}^m), [\mathbb{S}^m] \rangle = \chi(\mathbb{S}^m) = 2;$$

see, for example, [8, Chapter 11]. Thus

$$|\deg(e_a \circ \chi)| = 2. \tag{12}$$

If H^+ were not transitive on $\mathbb{S}^{m-1}$, the orbit H^+a would be a proper compact subset of the sphere. Any map factoring through H^+ and then through e_a would therefore miss a point of $\mathbb{S}^{m-1}$ and have degree zero, contradicting (12). Hence H^+ is transitive. $\square$

Theorem 5.2. *Let $m \geq 2$ be even, and let $K \subset \mathbb{R}^{m+1}$ be a star body whose central hyperplane sections are mutually affinely equivalent. Then every central hyperplane section of K is an ellipsoid.*

Proof. For each $u \in \mathbb{S}^m$ let

$$C_u = K \cap u^\perp$$

and normalize it intrinsically in the Euclidean space $u^\perp$ by

$$\widehat{C}_u = \mathrm{Cov}(C_u)^{-1/2}(C_u - b(C_u)).$$

By (3), the $\widehat{C}_u$ are mutually orthogonally congruent. Fix a normalized model $\widehat{S} \subset E$ and set

$$H = \{Q \in O(E) : Q\widehat{S} = \widehat{S}\}.$$

The bundle

$$\mathcal{R}_u = \{Q : E \to u^\perp : Q \text{ orthogonal and } Q\widehat{S} = \widehat{C}_u\}$$

is a principal H-subbundle of the orthonormal frame bundle of $T\mathbb{S}^m$. Local triviality follows exactly as in Proposition 2.3 from the continuity of $u \mapsto \widehat{C}_u$. Hence H is a reduction of the structure group of $T\mathbb{S}^m$.

Lemma 5.1 implies that an orientation-preserving subgroup of H acts transitively on $\mathbb{S}^{m-1}$. Consequently $\widehat{S}$ is invariant under a transitive subgroup of $SO(m)$, so membership in $\widehat{S}$ depends only on the Euclidean norm. Thus

$$\widehat{S} = \{x \in E : |x| \in I\}$$

for a compact set $I \subset [0, \infty)$. The body $\widehat{S}$ is an affine image of the original star body S, hence is homeomorphic to the closed m-ball; in particular it is connected and contractible, and it has nonempty interior. It follows that I is a nondegenerate compact interval $[a, b]$. If $a > 0$, then $\widehat{S}$ is a closed annulus and retracts onto $\mathbb{S}^{m-1}$, contradicting contractibility. Therefore $a = 0$ and

$$\widehat{S} = bB^m.$$

Undoing the centroid-covariance normalization shows that S, and hence every C_u, is an ellipsoid. $\square$

6 Patching ellipsoidal planar sections

The conclusion of Theorem 5.2 is enough to recover the ambient star body without first proving convexity. We record the elementary fact in a form that will also be useful for the final reduction.

Lemma 6.1 (planar patching)**.** *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be a star body. If every two-dimensional central section of K is an ellipse, then K is an ellipsoid.*

Proof. Write $\rho = \rho_K$. For $u \in \mathbb{S}^{n-1}$ define

$$q(u) = \frac{1}{\rho(u)\rho(-u)}, \qquad \lambda(u) = \frac{1}{\rho(u)} - \frac{1}{\rho(-u)}. \tag{13}$$

Extend these functions to $\mathbb{R}^n$ by

$$Q(ru) = r^2 q(u), \qquad L(ru) = r\lambda(u) \quad (r \geq 0),$$

with $Q(0) = L(0) = 0$. Thus Q is even and homogeneous of degree 2, while L is odd and homogeneous of degree 1.

Fix a two-dimensional subspace P. Since $K \cap P$ is an ellipse containing 0 in its interior, it has a normalized representation

$$K \cap P = \{x \in P : Q_P(x) + L_P(x) \leq 1\}, \tag{14}$$

where Q_P is a positive-definite quadratic form and L_P is linear. Indeed, expanding an equation $(x-c)^T A(x-c) \leq 1$ and dividing by $1 - c^T Ac > 0$ gives this form. For a unit vector $u \in P$, the two boundary points on the line $\mathbb{R}u$ give

$$Q_P(u)\rho(u)^2 + L_P(u)\rho(u) = 1,$$

$$Q_P(u)\rho(-u)^2 - L_P(u)\rho(-u) = 1.$$

Dividing respectively by $\rho(u)$ and $\rho(-u)$ and adding yields

$$Q_P(u) = \frac{1}{\rho(u)\rho(-u)} = q(u),$$

and substitution then gives

$$L_P(u) = \frac{1}{\rho(u)} - \frac{1}{\rho(-u)} = \lambda(u).$$

Thus

$$Q|_P = Q_P, \qquad L|_P = L_P. \tag{15}$$

Given arbitrary $x, y \in \mathbb{R}^n$, choose a two-plane containing them. From (15), Q satisfies the parallelogram identity on that plane, and hence globally. By polarization, Q is a positive-definite quadratic form. Likewise $L(x+y) = L(x) + L(y)$ on a two-plane containing x, y; together with continuity and homogeneity, this shows that L is a linear functional.

Every radial boundary point of K satisfies

$$Q(x) + L(x) = 1.$$

Conversely, because Q is positive definite and 0 satisfies the strict inequality, the set

$$\mathcal{E} = \{x \in \mathbb{R}^n : Q(x) + L(x) \leq 1\}$$

is an ellipsoid containing 0 in its interior. On each positive ray its boundary point is the same as that of K, by the construction above. Hence $K = \mathcal{E}$. $\square$

Corollary 6.2. *Under the hypotheses of Theorem 5.2, K is an ellipsoid.*

Proof. Every two-plane through the origin is contained in some central hyperplane. By Theorem 5.2, that hyperplane section is an ellipsoid, so its intersection with the two-plane is an ellipse. Lemma 6.1 applies. $\square$

7 Proof of Problem 7.4

We can now combine the two parities.

Proof of Theorem 1.1. Fix $V \in \mathrm{Gr}(n, k+1)$ and consider the star body $K_V = K \cap V$ in the $(k+1)$-dimensional Euclidean space V. Its central hyperplane sections are precisely

$$K \cap E, \qquad E \in \mathrm{Gr}(V, k),$$

and are therefore mutually affinely equivalent by hypothesis.

If k is odd, Theorem 4.2 shows that K_V is an ellipsoid. If k is even, Corollary 6.2 gives the same conclusion. Thus

$$K \cap V \text{ is an ellipsoid for every } V \in \mathrm{Gr}(n, k+1). \tag{16}$$

Every two-dimensional subspace $P \subset \mathbb{R}^n$ is contained in some $(k+1)$-dimensional subspace V. By (16),

$$K \cap P = (K \cap V) \cap P$$

is an ellipse. Lemma 6.1 now implies that K is an ellipsoid. □

8 Comments on the ingredients and checks

Remark 8.1 (where the recent theorem enters)*.* The only step above that uses the 2026 solution of Banach's isometric conjecture is the last step of Theorem 4.2, after convexity and central symmetry about the origin have already been obtained. Lu and Yang state the needed convex-geometric form as [6, Theorem 1.2]: an origin-symmetric convex body whose fixed-dimensional central sections are all linearly equivalent is an ellipsoid. Their preprint announces the remaining odd-dimensional cases and, together with Gromov's theorem, the full real Banach conjecture.

Remark 8.2 (why no Lipschitz hypothesis is needed)*.* Recent star-body arguments such as [1] obtain global exact families with Lipschitz dependence and then use Jacobian and Brouwer-degree identities. Here only the continuous principal bundle is needed. In odd section dimension, after a nonzero-degree pullback, Lemma 3.1 forces convexity before any differentiation is required. In even section dimension, the Euler-class argument is entirely topological. Thus continuity of the radial function suffices throughout.

Remark 8.3 (relation with the classical convex reduction)*.* For a convex body with affinely equivalent central hyperplane sections, Montejano's theorem gives central symmetry of every section, and Larman's False Center Theorem gives the dichotomy "ellipsoid or origin-symmetric." This is exactly the reduction needed to turn affine equivalence into linear equivalence before invoking Lu–Yang. Montejano states these two ingredients as Theorems 3.2 and 3.3 of [10].

Remark 8.4 (novelty check)*.* As of September 10, 2026, the publicly available update to *Geometric Tomography* still records the nonconvex star-body version of Problem 7.4 as open, while the August 2026 papers [6, 1] address the real linear Banach problem and related star-body linear-section questions. We are not aware of a prior argument using Lemma 3.1 or of a published solution of Problem 7.4 in the full affine star-body form.

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