Generalized intersection bodies and radial sums

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Abstract

The notion of a generalized intersection body was introduced by Zhang. Problem 8.5 in Gardner's *Geometric Tomography* asks whether, if L is a generalized intersection body in $\mathbb{R}^n$, there are intersection bodies L_1, L_2 such that L_1 is the radial sum of L and L_2. We show that the answer is negative in every dimension $n \geq 3$. The counterexample is generated by the even zonal signed density

$$q(v) = |v \cdot e_n|^{-1/2} \sin\bigl(|v \cdot e_n|^{-4}\bigr).$$

Although q is integrable, it has increasingly rapid sign oscillation near the equator. The reduction of the spherical Radon transform of a zonal function to a one-dimensional fractional integral is standard; see Rubin [4, (5.1.42)]. We use this reduction together with a direct oscillatory estimate. The spherical Radon transform of q is continuous because of cancellation, whereas the spherical Radon transform of the negative part of $M + \varepsilon q$ is unbounded near the poles for every fixed $M, \varepsilon > 0$. Adding a sufficiently large constant to q therefore produces the radial function of a generalized intersection body, while positivity and injectivity of the spherical Radon transform rule out every representation of this body as a radial difference of two intersection bodies. We also note that in dimension two the corresponding question has an affirmative answer.

1 Definitions and statement of the result

Let $n \geq 3$. A star body K in $\mathbb{R}^n$ has a positive continuous radial function ρ_K on S^{n-1}. For star bodies K, L, their radial sum $K \mathbin{\widetilde{+}} L$ is defined by

$$\rho_{K \widetilde{+} L} = \rho_K + \rho_L. \tag{1}$$

We use the unnormalized spherical Radon transform

$$(Rf)(u) = \int_{S^{n-1} \cap u^\perp} f(v)\, d\sigma_{u^\perp}(v), \qquad u \in S^{n-1}, \tag{2}$$

where $d\sigma_{u^\perp}$ is the usual surface measure on the $(n-2)$-sphere $S^{n-1} \cap u^\perp$. Thus

$$R1 = |S^{n-2}|. \tag{3}$$

For a finite signed Borel measure μ on S^{n-1}, $R\mu$ is defined distributionally by self-adjointness:

$$\langle R\mu, \varphi \rangle = \int_{S^{n-1}} (R\varphi)(v)\, d\mu(v), \qquad \varphi \in C^\infty(S^{n-1}). \tag{4}$$

If $d\mu = f\, d\sigma$ with $f \in L^1(S^{n-1})$, this agrees almost everywhere with the slice integral (2).

For background on the spherical Radon transform and its relation to one-dimensional fractional integration, we shall also use Rubin [4]. In particular, his formula (5.1.42) gives the standard fractional-integral representation for the transform of a zonal function. His Section 2.3 develops the Riemann–Liouville fractional integrals that underlie the Abel-type estimates used below.

The concept of a *generalized intersection body* was introduced by Zhang [2, p. 783]. We shall use the equivalent formulation given by Gardner on p. 340 of [1]: a star body L with continuous radial function is a generalized intersection body if

$$\rho_L = R\mu \tag{5}$$

for a finite signed even Borel measure μ on S^{n-1}. In Zhang's formulation, $R^{-1}\rho_L$ is called the dual generating distribution and L is a generalized intersection body when this distribution is a signed measure. Since R is bijective on even distributions, the two formulations are equivalent.

Similarly, an origin-symmetric star body K is an intersection body precisely when

$$\rho_K = R\nu \tag{6}$$

for a finite nonnegative even Borel measure ν; see Gardner [1, Chapter 8], Zhang [2, pp. 782–783], and Goodey–Weil [3]. Zhang also records, following Helgason, that R is a continuous bijection on even smooth functions and on even distributions [2, pp. 782–783]. In particular,

$$R\eta = 0, \quad \eta \text{ an even finite signed measure} \quad \Longrightarrow \quad \eta = 0. \tag{7}$$

Problem 8.5 asks whether for every generalized intersection body L there are intersection bodies L_1, L_2 such that

$$L_1 = L \,\widetilde{+}\, L_2.$$

Boris Rubin observed (private communication) that, if $\rho_L = R\mu$ and the two measures in the Jordan decomposition of μ both have continuous spherical Radon transforms, then the answer is affirmative for that L. The construction below shows that the opposite phenomenon can occur strongly enough to prevent *every* possible decomposition.

Theorem 1.1. *For every $n \geq 3$, there exists an origin-symmetric generalized intersection body L in $\mathbb{R}^n$ for which there do not exist intersection bodies L_1, L_2 satisfying*

$$L_1 = L \,\widetilde{+}\, L_2.$$

Consequently Problem 8.5 has a negative answer in every dimension $n \geq 3$.

2 The spherical Radon transform of a zonal function

Fix $e = e_n = (0, \ldots, 0, 1)$. Suppose that f is even and zonal about the e-axis,

$$f(v) = F(|v \cdot e|),$$

where F is measurable on $[0, 1]$. For $u \in S^{n-1}$ put

$$s = |u \cdot e|, \qquad a = (1 - s^2)^{1/2}. \tag{8}$$

If $a > 0$, the orthogonal projection of e onto $u^\perp$ has length a. Choose a unit vector $w \in u^\perp$ in the direction of this projection. If $v \in S^{n-1} \cap u^\perp$, then

$$v \cdot e = a(v \cdot w).$$

Using $r = v \cdot w$ as a coordinate on $S^{n-2} = S^{n-1} \cap u^{\perp}$ gives the standard slice formula

$$Rf(u) = 2|S^{n-3}| \int_0^1 F(ar)(1-r^2)^{(n-4)/2}\, dr. \tag{9}$$

Indeed, the surface measure of the slice with $v \cdot w = r$ is $|S^{n-3}|(1-r^2)^{(n-4)/2}\, dr$, and evenness supplies the factor 2. Notice that (9) also gives (3), since

$$2|S^{n-3}| \int_0^1 (1-r^2)^{(n-4)/2}\, dr = |S^{n-2}|.$$

For $n = 3$, (9) is the familiar Abel formula

$$Rf(u) = 4 \int_0^1 \frac{F(ar)}{\sqrt{1-r^2}}\, dr.$$

Formula (9) is a special case of the general one-dimensional fractional-integral representation of the spherical Radon transform on zonal functions; compare Rubin [4, (5.1.42)]. In Rubin's notation the right-hand side is expressed through an Erdélyi–Kober-type fractional integral. After a quadratic change of variable it becomes a Riemann–Liouville (Abel) fractional integral; see [4, Section 2.3] for the latter operators. We retain the elementary formula (9), since it keeps all constants and endpoint behavior transparent for the present example.

3 An oscillatory signed density

For $0 < t \le 1$ define

$$q_0(t) = t^{-1/2} \sin(t^{-4}), \tag{10}$$

and set $q_0(0) = 0$. On S^{n-1} put

$$q(v) = q_0(|v \cdot e|). \tag{11}$$

Because $t^{-1/2}$ is integrable at 0, the standard zonal integration formula on S^{n-1} shows that

$$q \in L^1(S^{n-1}).$$

Thus $q\, d\sigma$ is a finite signed even Borel measure.

We next prove that Rq nevertheless has a continuous representative. Let

$$x = a^2 = 1 - s^2, \qquad \lambda = a^{-4} = x^{-2}. \tag{12}$$

By (9), for $a > 0$,

$$Rq(u) = 2|S^{n-3}| a^{-1/2} J_n(\lambda) = 2|S^{n-3}| x^{-1/4} J_n(x^{-2}), \tag{13}$$

where

$$J_n(\lambda) = \int_0^1 r^{-1/2}(1-r^2)^{(n-4)/2} \sin(\lambda r^{-4})\, dr. \tag{14}$$

The decay in (13) is an instance of the familiar principle that oscillation in the density of a fractional integral produces decay in the exterior variable. Rubin develops this point for Riemann–Liouville fractional integrals on $(0, \infty)$ in [4, Section 2.3]. A change of variables such as $s = 1 - 1/(r+1)$ transfers statements between $(0,1)$ and $(0,\infty)$. In the present example it is simpler, and sharper for our purposes, to prove the required decay directly.

Lemma 3.1. *As $\lambda \to \infty$,*

$$J_3(\lambda) = O(\lambda^{-1/2}), \qquad J_n(\lambda) = O(\lambda^{-1}) \quad (n \geq 4).$$

Proof. Make the substitution $t = r^{-4}$. Then

$$J_n(\lambda) = \frac{1}{4} \int_1^\infty t^{-9/8}(1 - t^{-1/2})^{(n-4)/2} \sin(\lambda t)\, dt. \tag{15}$$

Write

$$b_n(t) = t^{-9/8}(1 - t^{-1/2})^{(n-4)/2}.$$

At infinity, $b_n(t) = O(t^{-9/8})$ and $b_n'(t) = O(t^{-17/8})$, so the contribution of $[2, \infty)$ is $O(\lambda^{-1})$ by integration by parts.

For $n = 3$, near $t = 1$ we have

$$b_3(t) = (t-1)^{-1/2} c(t),$$

where $c \in C^1[1, 2]$. Substitute $t = 1 + y^2$. The integral over $[1, 2]$ becomes the imaginary part of

$$e^{i\lambda} \int_0^1 A(y) e^{i\lambda y^2}\, dy$$

for some $A \in C^1[0, 1]$. This is $O(\lambda^{-1/2})$. For completeness, split at $\delta = \lambda^{-1/2}$. The integral over $[0, \delta]$ is $O(\delta)$. On $[\delta, 1]$ use

$$e^{i\lambda y^2} = \frac{1}{2i\lambda y} \frac{d}{dy} e^{i\lambda y^2}$$

and integrate by parts. The boundary terms are $O((\lambda\delta)^{-1}) = O(\lambda^{-1/2})$, while the resulting integral is bounded by

$$C\lambda^{-1} \left(\int_\delta^1 \frac{dy}{y} + \int_\delta^1 \frac{dy}{y^2} \right) = O(\lambda^{-1} \log \lambda) + O(\lambda^{-1/2}).$$

Hence $J_3(\lambda) = O(\lambda^{-1/2})$.

For $n = 4$, $b_4(t) = t^{-9/8}$ is continuously differentiable on $[1, \infty)$ with integrable derivative, so one integration by parts in (15) gives $O(\lambda^{-1})$.

Finally suppose $n \geq 5$. Put $\beta = (n-4)/2 > 0$. Near $t = 1$,

$$b_n(t) = O((t-1)^\beta), \qquad b_n'(t) = O((t-1)^{\beta - 1}).$$

Since $\beta > 0$, b_n' is integrable at 1; it is also integrable at infinity. Thus b_n is absolutely continuous on every finite interval $[1, T]$, with $b_n' \in L^1(1, \infty)$, and $b_n(1) = 0$. A single integration by parts over $[1, T]$, followed by $T \to \infty$, therefore gives $J_n(\lambda) = O(\lambda^{-1})$. $\square$

Proposition 3.2. *The distribution Rq is represented by a continuous even function on S^{n-1}, and this function vanishes at the two poles $\pm e$. More precisely, as $x = 1 - |u \cdot e|^2 \downarrow 0$,*

$$Rq(u) = \begin{cases} O(x^{3/4}), & n = 3, \\ O(x^{7/4}), & n \geq 4. \end{cases}$$

Proof. Combine (13) with Lemma 3.1 and $\lambda = x^{-2}$. For $n = 3$,

$$x^{-1/4}\lambda^{-1/2} = x^{-1/4}x = x^{3/4},$$

whereas for $n \geq 4$,

$$x^{-1/4}\lambda^{-1} = x^{-1/4}x^2 = x^{7/4}.$$

Thus the pointwise slice integral, defined for $a > 0$, tends to 0 at both poles.

It remains to check continuity away from the poles. On a set where $a \geq a_0 > 0$, the integrand in (9) for $f = q$ is bounded in absolute value by

$$Cr^{-1/2}(1 - r^2)^{(n-4)/2},$$

which is integrable on $(0, 1)$ for every $n \geq 3$. For each fixed $r > 0$ the integrand depends continuously on a. Dominated convergence therefore gives continuity. Since the slice formula agrees almost everywhere with the distributional transform of $q\,d\sigma$, the continuous representative just obtained represents Rq distributionally as well. $\square$

4 The negative part has an unbounded Radon transform

Fix $M > 0$ and $\varepsilon > 0$, and define

$$p(v) = M + \varepsilon q(v), \qquad p_-(v) = \max\{-p(v), 0\}. \tag{16}$$

The cancellation responsible for Proposition 3.2 disappears after taking the negative part. We first record an elementary averaging statement in the form needed below.

Lemma 4.1 (Periodic averaging). *Let P be a bounded 2π-periodic Riemann-integrable function with mean*

$$m = \frac{1}{2\pi}\int_0^{2\pi} P(t)\,dt.$$

If w is continuous on a compact interval $[\alpha, \beta]$, then

$$\int_\alpha^\beta w(r)P(\lambda r)\,dr \longrightarrow m\int_\alpha^\beta w(r)\,dr \qquad (\lambda \to \infty).$$

Proof. Partition $[\alpha, \beta]$ into complete intervals of length $2\pi/\lambda$, together with at most two remainders. On each complete interval replace w by its value at one endpoint. The error is bounded by the modulus of continuity of w at scale $2\pi/\lambda$, times a constant independent of λ. The two remainders have total length $O(\lambda^{-1})$. The contribution from the constant approximations on the complete periods is exactly the mean m times the corresponding Riemann sum for w. This proves the assertion. $\square$

Proposition 4.2. *There are constants $c > 0$ and $a_0 > 0$, depending on n, M, ε, such that*

$$Rp_-(u) \geq c\,a^{-1/2} = c(1 - |u\cdot e|^2)^{-1/4} \tag{17}$$

whenever $0 < a = (1 - |u\cdot e|^2)^{1/2} < a_0$. In particular,

$$Rp_-(u) \longrightarrow +\infty \qquad \text{as } u \to e \text{ or } u \to -e.$$

Proof. By (9),

$$Rp_-(u) = 2|S^{n-3}| \int_0^1 \left[M + \varepsilon a^{-1/2} r^{-1/2} \sin(a^{-4} r^{-4})\right]_- (1-r^2)^{(n-4)/2}\, dr, \tag{18}$$

where $[z]_- := \max\{-z, 0\}$. Restrict the integral to

$$I = [1/2, 3/4].$$

On I the factors $r^{-1/2}$ and $(1-r^2)^{(n-4)/2}$ are bounded above and below by positive constants. If

$$\sin(a^{-4} r^{-4}) \leq -\frac{1}{2},$$

then, for all sufficiently small a (uniformly in $r \in I$),

$$\left[M + \varepsilon a^{-1/2} r^{-1/2} \sin(a^{-4} r^{-4})\right]_- \geq \frac{\varepsilon}{4} a^{-1/2} r^{-1/2}. \tag{19}$$

Consequently

$$Rp_-(u) \geq c_1 a^{-1/2} \int_I \mathbf{1}_{\{\sin(a^{-4} r^{-4}) \leq -1/2\}}\, dr. \tag{20}$$

Set $t = r^{-4}$. The fixed interval I is carried, with reversed orientation, onto

$$I' = [(4/3)^4, 16],$$

and $dr = \frac{1}{4} t^{-5/4}\, dt$ in absolute value. Thus the integral in (20) is

$$\frac{1}{4} \int_{I'} t^{-5/4} \mathbf{1}_{\{\sin(a^{-4} t) \leq -1/2\}}\, dt.$$

Apply Lemma 4.1 with

$$P(y) = \mathbf{1}_{\{\sin y \leq -1/2\}}, \qquad w(t) = \frac{1}{4} t^{-5/4}.$$

The mean of P is positive (in fact 1/3). Hence the last integral tends to a strictly positive limit as $a \downarrow 0$, and is therefore bounded below by a positive constant for all sufficiently small a. Substitution into (20) proves (17). □

Remark 4.3. Taking $M = 0$ in Proposition 4.2 shows in particular that the spherical Radon transform of the negative part of $q\, d\sigma$ is unbounded, even though Rq is continuous. Thus continuity of the Radon transform of a signed measure does not imply continuity, or even boundedness, of the transforms of its Jordan parts.

5 Construction of the counterexample

We now prove Theorem 1.1. Fix $\varepsilon > 0$. By Proposition 3.2, Rq is a continuous function on the compact sphere. Choose $M > 0$ so large that

$$M|S^{n-2}| + \varepsilon Rq(u) > 0 \qquad \text{for all } u \in S^{n-1}. \tag{21}$$

Let

$$d\mu = p\, d\sigma = (M + \varepsilon q)\, d\sigma \tag{22}$$

and define

$$\rho_L = R\mu = M|S^{n-2}| + \varepsilon Rq. \tag{23}$$

The function ρ_L is positive, continuous, and even, so it is the radial function of an origin-symmetric star body L. Since μ is a finite signed even Borel measure, Gardner's definition [1, p. 340] shows directly that L is a generalized intersection body. Equivalently, in Zhang's terminology, $\mu = R^{-1}\rho_L$ is the dual generating signed measure [2, p. 783].

Assume, for contradiction, that there are intersection bodies L_1, L_2 such that

$$L_1 = L \,\widetilde{+}\, L_2. \tag{24}$$

Then

$$\rho_{L_1} = \rho_L + \rho_{L_2}. \tag{25}$$

Choose finite nonnegative even Borel measures ν_1, ν_2 representing the two intersection bodies:

$$\rho_{L_j} = R\nu_j, \qquad j = 1, 2. \tag{26}$$

Using (23) in (25), we get

$$R(\nu_1 - \nu_2 - \mu) = 0.$$

The measure in parentheses is even, so injectivity (7) gives

$$\nu_1 = \nu_2 + \mu. \tag{27}$$

Write the Jordan decomposition as

$$\mu = \mu^+ - \mu^-.$$

Since $\nu_1 = \nu_2 + \mu$ is nonnegative, we claim that

$$\nu_2 \geq \mu^-. \tag{28}$$

Indeed, let N be a negative Hahn set for μ. Then μ^+ is concentrated on $S^{n-1} \setminus N$ and μ^- on N. For every Borel set $A \subset N$, (27) yields

$$0 \leq \nu_1(A) = \nu_2(A) - \mu^-(A),$$

so $\nu_2(A) \geq \mu^-(A)$. Since μ^- vanishes off N, this proves (28) for every Borel set.

In our absolutely continuous example,

$$d\mu^- = p_-\, d\sigma. \tag{29}$$

Because the spherical Radon transform is positive, (28) implies

$$R\nu_2 \geq R\mu^- \tag{30}$$

in the sense of distributions. But $R\nu_2 = \rho_{L_2}$ is continuous and hence bounded on S^{n-1}; write

$$C = \max_{S^{n-1}} \rho_{L_2} < \infty.$$

On the other hand, Proposition 4.2 and (29) show that

$$R\mu^- = Rp_- > C + 1$$

whenever $0 < a = (1 - |u \cdot e|^2)^{1/2}$ is sufficiently small. For every fixed $a > 0$ the integral defining Rp_- is finite, and on compact sets away from the poles it depends continuously on u by dominated convergence. Choose a nonzero nonnegative test function $\varphi \in C_c^\infty$ supported in a sufficiently small annular polar region

$$0 < a_1 < (1 - |u \cdot e|^2)^{1/2} < a_2,$$

with a_2 small enough that $R\mu^- > C + 1$ throughout that region. From (30),

$$\int_{S^{n-1}} \rho_{L_2} \varphi \, d\sigma \geq \int_{S^{n-1}} (R\mu^-) \varphi \, d\sigma.$$

The left-hand side is at most $C \int \varphi \, d\sigma$, while the right-hand side is larger than $(C+1) \int \varphi \, d\sigma$, a contradiction. This proves Theorem 1.1.

6 Comments and the two-dimensional case

Remark 6.1 (Why the example works). The density $q_0(t) = t^{-1/2} \sin(t^{-4})$ combines an integrable singular amplitude with oscillations whose frequency tends to infinity at the equator. As the normal u approaches a pole, the spherical Radon transform samples these oscillations at frequency a^{-4}. For the signed density, cancellation forces $Rq(u) \to 0$. Taking the negative part destroys the cancellation while leaving amplitude of order $a^{-1/2}$ on a fixed positive proportion of a suitable annulus in the integration variable. This is the source of the blow-up in Proposition 4.2. The interpretation in terms of one-dimensional fractional integrals is consistent with Rubin's general treatment of zonal Radon-type transforms and oscillatory cancellation [4, Section 2.3 and (5.1.42)].

Remark 6.2 (An observation of Rubin). Boris Rubin observed (private communication) that if $\rho_L = R\mu$ and both Jordan parts of μ have continuous spherical Radon transforms, then L is a radial summand of an intersection body. Remark 4.3 shows that this regularity can fail drastically. More importantly, the domination argument (28) shows that replacing the Jordan decomposition by some other pair of positive representing measures cannot repair the failure for the body constructed here.

Remark 6.3 (Dimension two). The restriction $n \geq 3$ is natural. When $n = 2$, $S^1 \cap u^\perp$ consists of the two antipodal points $\{v, -v\}$ perpendicular to u. Hence, on even functions or measures, the spherical Radon transform is, up to the fixed factor 2, simply rotation through $\pi/2$. Its inverse therefore preserves positivity. Consequently every generalized intersection body in $\mathbb{R}^2$ is already an intersection body. Since the class of intersection bodies is closed under radial addition (representing positive measures simply add), one may choose, for example, a Euclidean ball L_2 and set $L_1 = L \mathbin{\widetilde{+}} L_2$. Thus Problem 8.5 has an affirmative answer in dimension two.

Remark 6.4 (No convexity assumption). No convexity is used or required. Zhang introduced generalized intersection bodies in the class of centered bodies, i.e. centrally symmetric star bodies [2, pp. 778–779, 783], and Gardner's equivalent formulation on p. 340 is likewise stated for star bodies. The counterexamples above need not be convex.

References

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