Convex Bodies of Constant i-Girth Without Constant i-Brightness

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Abstract

In 1970 Firey asked whether, for $1 < i < n-1$, there are convex bodies in $\mathbb{R}^n$ of constant i-girth which do not have constant outer i-measure (constant i-brightness). The question was later recorded as Problem 3.7 in Gardner's *Geometric Tomography.* We give an affirmative answer for every admissible pair (n, i). The examples may be chosen arbitrarily close to the Euclidean ball and with $C^{2,\alpha}$ support functions, for any fixed $0 < \alpha < 1$. The proof is local. An odd cubic perturbation of the ball is corrected, by the Banach-space implicit function theorem, so that its ith area measure has constant even part. This is exactly the condition of constant i-girth. A direct second-order calculation then shows that two i-dimensional projection volumes are unequal.

1 Introduction

Firey introduced the terminology *constant outer p-measure* for a convex body $K \subset \mathbb{R}^n$ whose orthogonal projections onto all p-dimensional linear subspaces have the same p-dimensional volume [3]. Thus constant outer p-measure is what is now usually called constant p-brightness. Firey also considered the corresponding lower-dimensional girth condition and asked, in the intermediate range

$$1 < p < n - 1, \tag{1.1}$$

whether constant p-girth can hold without constant outer p-measure. Gardner records this question, with attribution to Firey, as Problem 3.7 and discusses its history in Note 3.3 of [4].

Our purpose is to answer Firey's question affirmatively.

Theorem 1. *Let $n \geq 4$, let $2 \leq i \leq n-2$, and fix $0 < \alpha < 1$. There is a convex body $K \subset \mathbb{R}^n$, with support function of class $C^{2,\alpha}$ and everywhere positive principal radii of curvature, such that K has constant i-girth but not constant i-brightness. Such bodies can be chosen arbitrarily close to the Euclidean unit ball in $C^{2,\alpha}(S^{n-1})$.*

The proof uses only local differential geometry of support functions, the cosine-transform representation of girth, and the Banach-space implicit function theorem. Because the question concerns a rather classical part of convex geometry, we include the details of the analytic ingredients and all computations needed for the construction.

2 Notation and preliminaries

Throughout, S^{n-1} is the Euclidean unit sphere and $G(n, i)$ is the Grassmannian of i-dimensional linear subspaces of $\mathbb{R}^n$. If L is a linear subspace, $K|L$ denotes the orthogonal projection of K

onto L. We write V_j for the jth intrinsic volume, normalized so that V_j agrees with j-dimensional volume on j-dimensional convex bodies. These conventions agree with Schneider [8, Chs. 4–5].

2.1 Brightness and girth

Definition 2. *For $K \subset \mathbb{R}^n$ convex and $1 \le i \le n-1$, the* i-brightness function *is*

$$L \longmapsto V_i(K|L), \qquad L \in G(n,i).$$

The body has constant i-brightness *if this function is constant. Its* i-girth function *is*

$$u \longmapsto V_i(K|u^{\perp}), \qquad u \in S^{n-1},$$

and K has constant i-girth *if this function is constant.*

The implication "constant i-brightness $\Rightarrow$ constant i-girth" follows at once from Kubota's integral recursion; see Schneider [8, (5.3.23) and the surrounding discussion]. The problem is whether the converse fails.

2.2 Support functions and the matrix $Q(h)$

The support function of a convex body K is

$$h_K(u) = \max\{\langle x, u\rangle : x \in K\}, \qquad u \in S^{n-1}.$$

Let $h \in C^2(S^{n-1})$. Its spherical Hessian $\nabla_S^2 h$ is the covariant Hessian for the standard metric on the sphere. We put

$$Q(h) = \nabla_S^2 h + hI, \tag{2.1}$$

regarded at each u as a self-adjoint endomorphism of the tangent space $T_u S^{n-1}$.

If $h = h_K$ and K is C^2_+, the eigenvalues of $Q(h)(u)$ are the principal radii of curvature of ∂K at the point whose outer unit normal is u. Conversely, if $h \in C^2(S^{n-1})$ and $Q(h)$ is positive definite everywhere, then h is the support function of a strictly convex body with positive principal radii. These facts, including the parametrization

$$x(u) = h(u)u + \nabla_S h(u), \tag{2.2}$$

are proved in Schneider [8, §2.5, especially the discussion of reverse spherical images and support functions]. In the present proof we use this only near $h \equiv 1$: if $\|h-1\|_{C^2}$ is sufficiently small, then $Q(h)$ is a small perturbation of I, hence is positive definite.

2.3 Area measures

For $1 \le i \le n-1$, $S_i(K,\cdot)$ denotes the ith area measure of K, in Schneider's normalization [8, §4.2 and §5.1]. If $h = h_K \in C^2$ and $Q(h) > 0$, then $S_i(K,\cdot)$ has a density with respect to spherical Lebesgue measure. If $d = n-1$, that density, divided by its value for the unit ball, is

$$\sigma_i(h) = \binom{d}{i}^{-1} e_i(Q(h)), \tag{2.3}$$

where $e_i(A)$ is the ith elementary symmetric function of the d eigenvalues of A. Formula (2.3) is the smooth form of the mixed-area-measure formula; see Schneider [8, §5.1, in particular the mixed discriminant representation]. The same Hessian equation is written explicitly in Guan–Ma [5, (1.1)–(1.2)].

The normalization in (2.3) will be convenient because

$$\sigma_i(1) = 1. \tag{2.4}$$

2.4 The cosine transform and constant girth

For a finite signed Borel measure μ on S^{n-1}, its cosine transform is

$$(C\mu)(u) = \int_{S^{n-1}} |\langle u, v\rangle| \, d\mu(v). \tag{2.5}$$

Kiderlen gives the following representation of the i-girth function:

$$V_i(K|u^\perp) = \frac{1}{2}(CS_i(K, \cdot))(u), \tag{2.6}$$

with the area-measure and intrinsic-volume conventions used there; see Kiderlen [6, p. 2014, subsection "Girth functions and projection functions"]. A change of normalization changes only the common positive factor and is immaterial here.

For a measure μ, define its even part by

$$\mu_{\mathrm{ev}}(A) = \frac{1}{2}\{\mu(A) + \mu(-A)\}.$$

Since the kernel $|\langle u, v\rangle|$ is even in v, $C\mu = C\mu_{\mathrm{ev}}$. The cosine transform is injective on even finite signed measures. One proof expands in spherical harmonics: its multipliers in every even degree are nonzero. Kiderlen gives this argument explicitly in [6, pp. 2001–2002]; see also Schneider [8, Appendix, spherical harmonics and the cosine transform].

It follows from (2.6) that, for a C^2_+ body,

$$\boxed{K \text{ has constant } i\text{-girth} \iff \text{the even part of } \sigma_i(h_K) \text{ is constant.}} \tag{2.7}$$

Indeed, if the even part of σ_i is constant, (2.6) is constant by rotational invariance. Conversely, if (2.6) is constant, subtract a suitable constant multiple of spherical measure; the resulting even signed measure has zero cosine transform and hence is zero.

2.5 Hölder spaces and the analytic tool

For $0 < \alpha < 1$, $C^{k,\alpha}(S^{n-1})$ is the usual Hölder space: derivatives through order k are continuous and the kth derivatives are Hölder continuous with exponent α. We use the closed subspace

$$C^{k,\alpha}_{\mathrm{ev},0} = \left\{ f \in C^{k,\alpha}(S^{n-1}) : f(-u) = f(u), \quad \int_{S^{n-1}} f = 0 \right\}. \tag{2.8}$$

We shall use the Banach-space implicit function theorem in its standard form: if $F : X \times Y \to Z$ is continuously differentiable, $F(x_0, y_0) = 0$, and $D_y F(x_0, y_0) : Y \to Z$ is a bounded linear isomorphism, then near x_0 there is a unique continuously differentiable function $y = y(x)$ with $F(x, y(x)) = 0$. A proof may be found, for example, in Deimling [2, §15.1].

We also need the following elementary consequence of the elliptic theory of the sphere.

Lemma 3. *The operator*

$$\Delta_S + d : C^{2,\alpha}_{\mathrm{ev},0}(S^{n-1}) \longrightarrow C^{0,\alpha}_{\mathrm{ev},0}(S^{n-1}), \qquad d = n - 1, \tag{2.9}$$

is a bounded linear isomorphism.

Proof. The spherical harmonics of degree k are the restrictions to S^{n-1} of homogeneous harmonic polynomials of degree k. They satisfy

$$\Delta_S Y_k = -k(k+n-2)Y_k; \tag{2.10}$$

see Schneider [8, Appendix, spherical harmonics]. Therefore the eigenvalue of $\Delta_S + d$ on degree k is

$$d - k(k+d-1). \tag{2.11}$$

It vanishes exactly for $k = 1$. Degree-one harmonics are odd, so the kernel on even functions is zero. Constants are excluded only to make the range mean-zero: integrating $\Delta_S f = 0$ gives

$$\int (\Delta_S + d)f = d \int f.$$

It remains to justify existence with the stated Hölder regularity. The operator $\Delta_S + d$ is a uniformly elliptic second-order operator with smooth coefficients on the compact manifold S^{n-1}. The Fredholm alternative together with the global Schauder estimate says that a $C^{0,\alpha}$ right-hand side orthogonal to the kernel of the adjoint has a $C^{2,\alpha}$ solution, unique modulo the kernel. For this standard result on compact manifolds see Aubin [1, Ch. 4, §4, the Fredholm alternative for elliptic operators and Schauder estimates]. The operator is self-adjoint, and on the even subspace its kernel is zero by (2.11). Hence every even mean-zero right-hand side has a unique even mean-zero $C^{2,\alpha}$ solution. The Schauder estimate gives boundedness of the inverse. □

3 Imposing constant i-girth

Put

$$\psi(u) = u_1 u_2 u_3. \tag{3.1}$$

This is odd and is a spherical harmonic of degree 3, since the homogeneous polynomial $x_1x_2x_3$ is harmonic in $\mathbb{R}^n$.

For a function f let

$$f_{\mathrm{ev}}(u) = \frac{1}{2}\{f(u) + f(-u)\}, \qquad P_{\mathrm{ev},0} f = f_{\mathrm{ev}} - \frac{1}{|S^{n-1}|}\int_{S^{n-1}} f_{\mathrm{ev}}. \tag{3.2}$$

Define

$$\mathcal{F}(\varepsilon, \eta) = P_{\mathrm{ev},0}\sigma_i(1 + \varepsilon\psi + \eta), \tag{3.3}$$

for ε and $\eta \in C^{2,\alpha}_{\mathrm{ev},0}$ small enough that $Q(1 + \varepsilon\psi + \eta) > 0$.

The map $\mathcal{F}$ is continuously differentiable from a neighborhood of $(0, 0)$ in $\mathbb{R} \times C^{2,\alpha}_{\mathrm{ev},0}$ to $C^{0,\alpha}_{\mathrm{ev},0}$. This requires no nonlinear PDE theorem: $h \mapsto Q(h)$ is bounded linear from $C^{2,\alpha}$ to $C^{0,\alpha}$ matrix fields, e_i is a polynomial in the matrix entries, and $P_{\mathrm{ev},0}$ is bounded linear.

For a $d \times d$ matrix A,

$$\frac{d}{dt} e_i(I + tA)\Big|_{t=0} = \binom{d-1}{i-1} \operatorname{tr} A. \tag{3.4}$$

This follows directly by choosing the i eigenvalues in the definition of e_i, or by differentiating the principal-minor formula. Since

$$\operatorname{tr} Q(f) = \Delta_S f + df, \tag{3.5}$$

(2.3)–(3.5) give

$$D_\eta \mathcal{F}(0,0)\eta = \frac{i}{d}(\Delta_S + d)\eta. \tag{3.6}$$

Lemma 3 shows that (3.6) is an isomorphism.

The implicit function theorem therefore proves the following.

Lemma 4. *For all sufficiently small $|\varepsilon|$, there is a unique $\eta_\varepsilon \in C^{2,\alpha}_{\mathrm{ev},0}$, near zero, such that*

$$h_\varepsilon = 1 + \varepsilon\psi + \eta_\varepsilon \tag{3.7}$$

is the support function of a $C^{2,\alpha}_+$ convex body K_ε of constant i-girth. Moreover

$$\eta_0 = 0, \qquad \eta_0' = 0. \tag{3.8}$$

Proof. The existence and uniqueness follow from the preceding paragraph and the implicit function theorem. For small ε, $h_\varepsilon \to 1$ in $C^{2,\alpha}$; hence $Q(h_\varepsilon) > 0$, so h_ε is a support function by the result cited after (2.2). Equation $\mathcal{F}(\varepsilon, \eta_\varepsilon) = 0$ is exactly constant i-girth by (2.7).

To prove $\eta_0' = 0$, differentiate $\mathcal{F}(\varepsilon, \eta_\varepsilon) = 0$ at $\varepsilon = 0$. The derivative of $\sigma_i(1 + \varepsilon\psi)$ at zero is

$$\frac{i}{d}(\Delta_S + d)\psi,$$

which is odd because ψ is odd. The projection $P_{\mathrm{ev},0}$ therefore kills it. Thus $D_\eta\mathcal{F}(0,0)\eta_0' = 0$, and (3.6) is injective. $\square$

Because the implicit solution is C^2 as a function of the scalar parameter ε (the map $\mathcal{F}$ is in fact polynomial in its function argument), Taylor's theorem in Banach spaces gives

$$\eta_\varepsilon = \varepsilon^2\eta_2 + o(\varepsilon^2) \quad \text{in } C^{2,\alpha}, \tag{3.9}$$

where $\eta_2 = \frac{1}{2}\eta_0''$. This is the only regularity in the parameter that we shall need; it follows from the C^2 version of the same implicit-function theorem [2, §15.1].

4 Computing the quadratic correction

Set

$$A = \nabla_S^2\psi + \psi I. \tag{4.1}$$

For symmetric $d \times d$ matrices A, B, direct expansion of the principal minors gives

$$\begin{aligned} e_i(I + \varepsilon A + \varepsilon^2 B) = \binom{d}{i} &+ \varepsilon\binom{d-1}{i-1}\operatorname{tr} A \\ &+ \varepsilon^2\left\{\binom{d-1}{i-1}\operatorname{tr} B + \binom{d-2}{i-2}e_2(A)\right\} + o(\varepsilon^2). \end{aligned} \tag{4.2}$$

For completeness, the coefficient of $\operatorname{tr} B$ is obtained by choosing one B-entry and $i-1$ identity entries; the quadratic terms in A are obtained by choosing two A-entries and $i-2$ identity entries, and their sum is $e_2(A)$.

Substituting (3.9) into (2.3), taking the even mean-zero part, and using

$$\frac{\binom{d-2}{i-2}}{\binom{d-1}{i-1}} = \frac{i-1}{d-1},$$

we obtain

$$(\Delta_S + d)\eta_2 = -\frac{i-1}{n-2} e_2(A) + c, \tag{4.3}$$

where c is a constant chosen so that the right side has mean zero.

Put

$$s_1 = u_1^2 + u_2^2 + u_3^2, \quad s_2 = u_1^2 u_2^2 + u_1^2 u_3^2 + u_2^2 u_3^2, \quad p = u_1^2 u_2^2 u_3^2. \tag{4.4}$$

Lemma 5. *For $\psi = u_1 u_2 u_3$,*

$$e_2(A) = -s_1 + 4s_2 + 2(n-2)(n+5)p. \tag{4.5}$$

A solution of (4.3), modulo an additive constant, is

$$\eta_2 = \frac{i-1}{15(n-2)(n+1)} s_1 + \frac{4(i-1)}{15(n-2)} s_2 + \frac{2(i-1)}{5} p. \tag{4.6}$$

Proof. We give a coordinate verification of (4.5). Let $P(x) = x_1 x_2 x_3$, so that $\psi = P|_{S^{n-1}}$. For a homogeneous polynomial of degree m,

$$\Delta_S(P|_S) = (\Delta_{\mathbb{R}^n} P)|_S - m(m+n-2)P|_S. \tag{4.7}$$

This follows by writing the Euclidean Laplacian in polar coordinates:

$$\Delta_{\mathbb{R}^n} = \partial_{rr} + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\Delta_S.$$

Since P is harmonic and $m = 3$,

$$\Delta_S \psi = -3(n+1)\psi.$$

Consequently

$$\operatorname{tr} A = (\Delta_S + d)\psi = -2(n+2)\psi. \tag{4.8}$$

To compute $\operatorname{tr}(A^2)$, use the identity

$$\nabla_S^2 \psi(X,Y) = D^2 P(X,Y) - 3P(u)\langle X, Y\rangle \quad (X, Y \perp u), \tag{4.9}$$

which follows by differentiating $P|_S$ twice along the sphere and using homogeneity $DP(u)\cdot u = 3P(u)$. Hence

$$A(X,Y) = D^2 P(X,Y) - 2P(u)\langle X, Y\rangle. \tag{4.10}$$

The Euclidean Hessian of P has zero diagonal and

$$P_{12} = u_3, \qquad P_{13} = u_2, \qquad P_{23} = u_1. \tag{4.11}$$

Restricting (4.10) to $u^\perp$, squaring, and summing over an orthonormal basis of $u^\perp$ gives

$$\operatorname{tr}(A^2) = 2s_1 - 8s_2 + \{4(n+2)^2 - 4(n-2)(n+5)\}p. \tag{4.12}$$

For readers wishing to check (4.12) without choosing a tangent basis, let $H = D^2P(u)$ and $\Pi = I - u \otimes u$. The squared Hilbert–Schmidt norm of the restriction of H to $u^\perp$ is

$$\operatorname{tr}(\Pi H \Pi H) = \operatorname{tr}(H^2) - 2|Hu|^2 + (u^T H u)^2,$$

and Euler's identities give $Hu = 2\nabla P$ and $u^T H u = 6P$. Substituting (4.11) then yields (4.12) after elementary expansion.

Finally

$$e_2(A) = \frac{1}{2}\{(\operatorname{tr} A)^2 - \operatorname{tr}(A^2)\},$$

and (4.8), (4.12) give (4.5).

For (4.6), apply (4.7) to the homogeneous polynomials occurring in (4.4). Direct Euclidean differentiation gives

$$\begin{aligned}(\Delta_S + d)s_1 &= 6 - (n+1)s_1,\\ (\Delta_S + d)s_2 &= 4s_1 - 3(n+3)s_2,\\ (\Delta_S + d)p &= 2s_2 - 5(n+5)p.\end{aligned} \tag{4.13}$$

Substituting (4.6) into (4.13) gives

$$(\Delta_S + d)\eta_2 = -\frac{i-1}{n-2}e_2(A) + \frac{2(i-1)}{5(n-2)(n+1)},$$

which is (4.3). □

5 Second variation of the i-brightness

Let $L \in G(n,i)$. The support function of $K|L$, regarded as a convex body in L, is simply h_K restricted to $S^{n-1} \cap L$; this follows immediately from the definition: for $u \in L$,

$$h_{K|L}(u) = \max_{x\in K}\langle x|L, u\rangle = \max_{x\in K}\langle x, u\rangle = h_K(u). \tag{5.1}$$

For a C^2_+ convex body M in the i-dimensional Euclidean space L, with support function h,

$$V_i(M) = \frac{1}{i}\int_{S^{i-1}} h\,\det(\nabla_L^2 h + hI)\,dS. \tag{5.2}$$

This is the usual formula $V_i(M) = \frac{1}{i}\int h\,dS_{i-1}(M)$; see Schneider [8, (5.1.16) together with the smooth density formula in §4.2].

Lemma 6. *Let*

$$h_\varepsilon|_L = 1 + \varepsilon\psi + \varepsilon^2\eta_2 + o(\varepsilon^2) \quad \text{in } C^2(S^{n-1}\cap L).$$

Then

$$\begin{aligned}V_i(K_\varepsilon|L) = \kappa_i + \varepsilon^2\bigg\{&\int_{S^{n-1}\cap L}\eta_2\,dS\\ &+ \frac{1}{2}\int_{S^{n-1}\cap L}\psi(\Delta_L + i - 1)\psi\,dS\bigg\} + o(\varepsilon^2).\end{aligned} \tag{5.3}$$

Proof. Put $q = i-1$ and

$$B = \nabla_L^2\psi + \psi I, \qquad C = \nabla_L^2\eta_2 + \eta_2 I.$$

The elementary determinant expansion

$$\det(I + \varepsilon B + \varepsilon^2 C) = 1 + \varepsilon\operatorname{tr} B + \varepsilon^2\{\operatorname{tr} C + e_2(B)\} + o(\varepsilon^2) \tag{5.4}$$

inserted in (5.2) gives, after integration, the quadratic coefficient

$$\frac{1}{i}\int\{i\eta_2 + e_2(B) + \psi\operatorname{tr} B\}\,dS. \tag{5.5}$$

Here we used $\int \Delta_L \eta_2 = 0$.

We claim

$$\int e_2(B)\, dS = \frac{q-1}{2} \int \psi(\Delta_L + q)\psi\, dS. \tag{5.6}$$

To verify this, use

$$2e_2(B) = (\operatorname{tr} B)^2 - \operatorname{tr}(B^2)$$

and expand $B = \nabla^2\psi + \psi I$. The integrated Bochner formula on the unit q-sphere is

$$\int |\nabla^2\psi|^2 = \int (\Delta\psi)^2 - (q-1) \int |\nabla\psi|^2. \tag{5.7}$$

Formula (5.7) follows by integrating the Bochner identity

$$\frac{1}{2}\Delta|\nabla\psi|^2 = |\nabla^2\psi|^2 + \langle \nabla\psi, \nabla\Delta\psi\rangle + (q-1)|\nabla\psi|^2;$$

see, for example, Petersen [7, Ch. 7, Bochner formula]. Using also

$$\int \psi\Delta\psi = -\int |\nabla\psi|^2,$$

a direct simplification gives (5.6). Since $\operatorname{tr} B = (\Delta_L + q)\psi$, substitution of (5.6) into (5.5), with $i = q+1$, yields (5.3). The linear term is zero because ψ is odd and the sphere $S^{n-1} \cap L$ is centrally symmetric. □

We now compare two i-planes:

$$L_0 = \operatorname{span}\{e_1, e_2, e_4, \ldots, e_{i+1}\}, \tag{5.8}$$

and

$$L_1 = H \oplus \operatorname{span}\{e_4, \ldots, e_{i+1}\}, \qquad H = \{x \in \operatorname{span}(e_1, e_2, e_3) : x_1 + x_2 + x_3 = 0\}. \tag{5.9}$$

On L_0, $\psi = 0$. On H, the polynomial $x_1x_2x_3$ is harmonic with respect to the Euclidean structure of H. One quick verification is to use coordinates

$$x_1 = \frac{s}{\sqrt{2}} + \frac{t}{\sqrt{6}}, \quad x_2 = -\frac{s}{\sqrt{2}} + \frac{t}{\sqrt{6}}, \quad x_3 = -\frac{2t}{\sqrt{6}};$$

then $x_1x_2x_3$ is a constant multiple of $3s^2t - t^3$, whose two-dimensional Laplacian is zero. It remains harmonic after adding the unused coordinates $e_4, \ldots, e_{i+1}$. Therefore, on the unit sphere in L_1, (4.7) with dimension i gives

$$\Delta_L\psi = -3(i+1)\psi, \qquad (\Delta_L + i - 1)\psi = -2(i+2)\psi. \tag{5.10}$$

We need only elementary spherical moments.

Lemma 7. *Let U be uniformly distributed on the unit sphere of the indicated i-plane. On L_0,*

$$\mathbb{E}s_1 = \frac{2}{i}, \qquad \mathbb{E}s_2 = \frac{1}{i(i+2)}, \qquad \mathbb{E}p = 0. \tag{5.11}$$

On L_1,

$$\mathbb{E}s_1 = \frac{2}{i}, \qquad \mathbb{E}s_2 = \frac{2}{i(i+2)}, \qquad \mathbb{E}p = \frac{4}{9i(i+2)(i+4)}. \tag{5.12}$$

Proof. We first recall the moment formula. If U is uniform on $S^{i-1} \subset \mathbb{R}^i$, rotational invariance gives

$$\mathbb{E} \prod_{\nu=1}^{2m} \langle a_\nu, U \rangle = \frac{1}{i(i+2)\cdots(i+2m-2)} \sum_{\mathcal{P}} \prod_{\{r,s\} \in \mathcal{P}} \langle a_r, a_s \rangle, \tag{5.13}$$

where the sum is over all pairings of $\{1, \dots, 2m\}$. For $m = 1, 2, 3$, which are the only cases used here, (5.13) can be proved directly by rotational invariance and the identity $\sum U_j^2 = 1$; see Schneider–Weil [9, §13.2, spherical moment formulas].

For L_0, $U_3 = 0$, and (5.11) follows from

$$\mathbb{E} U_a^2 = \frac{1}{i}, \qquad \mathbb{E} U_a^2 U_b^2 = \frac{1}{i(i+2)} \quad (a \neq b).$$

For L_1, let P_H be orthogonal projection of $\mathbb{R}^3$ onto H. Its matrix is

$$P_H = I - \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

so its diagonal entries are $2/3$ and its off-diagonal entries are $-1/3$. Applying (5.13) to the three coordinate linear forms gives

$$\mathbb{E} U_a^2 = \frac{2}{3i},$$

$$\mathbb{E} U_a^2 U_b^2 = \frac{(2/3)^2 + 2(-1/3)^2}{i(i+2)} = \frac{2}{3i(i+2)} \quad (a \neq b).$$

Summing proves the first two formulas in (5.12).

For the sixth moment, (5.13) has fifteen pairings. In $\mathbb{E}(U_1^2 U_2^2 U_3^2)$, one pairing pairs equal coordinates in all three pairs, contributing $(2/3)^3 = 8/27$; six pairings contain exactly one equal-coordinate pair, contributing $(2/3)(-1/3)^2 = 2/27$ each; and eight pairings contain no equal-coordinate pair, contributing $(-1/3)^3 = -1/27$ each. The sum is

$$\frac{8 + 12 - 8}{27} = \frac{4}{9}.$$

Division by $i(i+2)(i+4)$ proves the last formula in (5.12). □

Proposition 8. *For the bodies K_ε of Lemma 4,*

$$V_i(K_\varepsilon | L_1) - V_i(K_\varepsilon | L_0) = |S^{i-1}| \frac{4(i-n+1)}{15\, i(i+2)(n-2)} \varepsilon^2 + o(\varepsilon^2). \tag{5.14}$$

Proof. We use normalized spherical averages and multiply by $|S^{i-1}|$ at the end. By (4.6), (5.11), and (5.12), the s_1-term has the same average on the two planes, while

$$\begin{aligned} \mathbb{E}_{L_1} \eta_2 - \mathbb{E}_{L_0} \eta_2 &= \frac{4(i-1)}{15(n-2)} \frac{1}{i(i+2)} + \frac{2(i-1)}{5} \frac{4}{9i(i+2)(i+4)} \\ &= \frac{4(i-1)(3i+2n+8)}{45\, i(i+2)(i+4)(n-2)}. \end{aligned} \tag{5.15}$$

On L_0, $\psi = 0$. On L_1, (5.10) and the last formula in (5.12) give

$$\frac{1}{2}\mathbb{E}\{\psi(\Delta_L + i - 1)\psi\} = -(i+2)\mathbb{E}p = -\frac{4}{9i(i+4)}. \tag{5.16}$$

Adding (5.15) and (5.16), putting over a common denominator, and cancelling $i+4$, gives

$$\frac{4(i-n+1)}{15\,i(i+2)(n-2)}. \tag{5.17}$$

Lemma 6 now yields (5.14). □

Proof of Theorem 1. For all sufficiently small ε, Lemma 4 gives a $C_+^{2,\alpha}$ body K_ε of exactly constant i-girth. Since $2 \le i \le n-2$,

$$i - n + 1 < 0.$$

The coefficient of ε^2 in (5.14) is therefore nonzero. Hence, for every sufficiently small nonzero ε,

$$V_i(K_\varepsilon|L_1) \neq V_i(K_\varepsilon|L_0).$$

Thus K_ε does not have constant i-brightness. Finally, $h_\varepsilon \to 1$ in $C^{2,\alpha}$, so the examples can be chosen arbitrarily close to the unit ball. □

6 Concluding remarks

Remark 9. *The construction is necessarily non-centrally-symmetric: if K is origin-symmetric, its ith area measure is even, so constant i-girth forces that area measure itself to be constant by (2.7). Our perturbation begins in an odd direction, precisely where the cosine transform loses information.*

Remark 10. *The proof establishes the existence of $C_+^{2,\alpha}$ examples for every $0 < \alpha < 1$, which is more than enough to answer Firey's question. We have deliberately made no C^∞ regularity claim: such a claim is unnecessary for the problem and would require an additional regularity argument for the nonlinear equation.*

Remark 11. *The decisive feature of the perturbation $\psi = u_1u_2u_3$ is that it is non-zonal. The constant-girth constraint determines an even quadratic correction, but the restriction of ψ to the plane L_1 remains a nonzero harmonic cubic. Formula (5.17) measures the surviving second-order variation of the i-brightness.*

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