Verification, Successive Determination, and Determination of Ellipsoids by k-Dimensional X-Rays

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Abstract

Problem 2.12 in Gardner's *Geometric Tomography* asks for the minimum number of k-dimensional X-rays needed to determine an n-dimensional ellipsoid among measurable sets. We distinguish three notions. In *verification*, the subspaces may depend on the ellipsoid. In *successive determination*, the first subspace is chosen independently of the unknown ellipsoid and each later subspace may depend on the X-rays already taken. In *determination*, one fixed family of subspaces must work for every ellipsoid. We solve all three problems. The verification number is

$$\left\lceil \frac{n}{n-k} \right\rceil.$$

Restricting the competing class from measurable sets to ellipsoids does not change any of the three answers. Successive determination affords no numerical saving over determination. For $n \geq 3$, their common number is

$$\begin{cases} 3, & k \leq (n+1)/3, \\ 4, & (n+1)/3 < k < n/2, \\ \left\lceil \dfrac{n(n+1)}{(n-k)(n-k+1)} \right\rceil, & k \geq n/2. \end{cases}$$

For $n = 2$ (necessarily $k = 1$), the number is 3. The proof reduces the determination problem to the injectivity of a restriction map on quadratic forms, and hence to the absence of a quadric containing a suitable union of projective linear subspaces. The required generic Hilbert-function calculation is supplied by a theorem of Carlini, Catalisano, and Geramita. A strict-concavity argument for the determinant then shows that allowing the X-ray subspaces to be selected successively does not reduce the number required.

1 Introduction

Let $1 \leq k \leq n-1$, and let S be a k-dimensional linear subspace of $\mathbb{R}^n$. If $E \subset \mathbb{R}^n$ is measurable, its k-dimensional X-ray parallel to S is the function on $S^\perp$ defined by

$$X_S E(y) = \mathcal{H}^k\big(E \cap (y + S)\big), \qquad y \in S^\perp,$$

whenever the sections are measurable. Here $\mathcal{H}^k$ denotes k-dimensional Lebesgue measure on the affine plane $y + S$. Sets that differ by a null set have the same X-rays almost everywhere (by Fubini's theorem), and uniqueness below is always understood modulo null sets. Accordingly, equality of X-rays is understood almost everywhere on the corresponding orthogonal complement.

Problem 2.12 of Gardner's book *Geometric Tomography* asks for the minimum number of k-dimensional X-rays required to determine an n-dimensional ellipsoid among measurable sets; see [6, Chapter 2, Problem 2.12]. The distinction among three natural notions is important.

In the *verification problem*, a particular ellipsoid is known in advance, and the X-ray subspaces are allowed to depend on that ellipsoid. In *successive determination*, the ellipsoid is not known in advance: the first X-ray is taken parallel to an arbitrarily prescribed k-dimensional subspace, and after each X-ray has been observed, its information may be used in choosing the next subspace. This interactive notion was introduced in geometric probing by Edelsbrunner and Skiena [2] and was developed for X-rays of polytopes by Gardner and Gritzmann [3]. In the *determination problem*, one seeks a fixed collection of k-dimensional subspaces, chosen independently of the ellipsoid, whose X-rays determine every ellipsoid.

For ordinary X-rays ($k = 1$), verification and determination already have different answers. Two suitably chosen directions verify any ellipsoid, whereas Gardner proved that every ellipsoid in $\mathbb{R}^n$ is unique for any set of three directions [5]. We shall see that, for ellipsoids, successive determination has the same numerical answer as determination, even though successive determination is a priori a more flexible procedure. We shall also see that none of the three numerical answers changes if the competing class is restricted from measurable sets to ellipsoids.

The main tools are elementary linear algebra, strict concavity of the log-determinant, the theory of additive sets developed by Fishburn–Lagarias–Reeds–Shepp [4] and Gardner [5], and a 2012 theorem of Carlini–Catalisano–Geramita on quadrics containing generic configurations of projective linear spaces [1].

We thank Kai-Wen Yang for communicating a paper produced by AI in which the verification problem is solved. The solution presented below in Theorem 3.1, which makes use of additivity, is shorter.

2 Additive sets and a uniqueness principle

We recall only the part of the additive-set theory that will be needed. Let $S_1, \ldots, S_m$ be linear subspaces of $\mathbb{R}^n$. A function f_i on $\mathbb{R}^n$ is a ridge function orthogonal to S_i if it is constant on every affine subspace parallel to S_i. Equivalently, if P_i is the orthogonal projection onto $S_i^\perp$, then

$$f_i(x) = g_i(P_i x)$$

for some function g_i on $S_i^\perp$.

A measurable set E is called additive with respect to $S_1, \ldots, S_m$ if, modulo a null set,

$$E = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{m} f_i(x) > 0 \right\}, \tag{2.1}$$

where each f_i is a ridge function orthogonal to S_i. Gardner's formulation allows either $>$ or $=$ in the defining relation and allows a finite set of subspaces, not merely one-dimensional directions [5].

The uniqueness principle we use is the following.

Theorem 2.1 (Additive-set uniqueness)**.** *If a measurable set E is additive with respect to $S_1, \ldots, S_m$, then E is determined, modulo null sets, by its X-rays parallel to $S_1, \ldots, S_m$.*

For coordinate projections this principle appears in Fishburn, Lagarias, Reeds, and Shepp [4]; Gardner developed the corresponding theory for arbitrary finite sets of directions and, more generally, subspaces [5]. See also [6, Chapter 2] for an exposition in the language of geometric tomography.

3 Verification of a single ellipsoid

We begin with the version in which the X-ray subspaces may depend on the ellipsoid.

Theorem 3.1 (Verification number)**.** *Let $1 \le k \le n-1$. The minimum number of k-dimensional X-rays needed to verify an arbitrary ellipsoid in $\mathbb{R}^n$ among measurable sets is*

$$\boxed{\left\lceil \frac{n}{n-k} \right\rceil}.$$

The same number is required if the competing sets are restricted to ellipsoids.

Proof. We first prove the lower bound. Suppose that the X-rays parallel to k-dimensional subspaces $S_1, \dots, S_m$ determine an ellipsoid E. Necessarily

$$S_1 \cap \cdots \cap S_m = \{0\}. \tag{3.1}$$

Indeed, if $0 \neq z$ belongs to the intersection, then E and $E+z$ have the same X-ray parallel to every S_i: translation by $z \in S_i$ merely translates points within each affine plane parallel to S_i.

Since $\operatorname{codim} S_i = n-k$,

$$\operatorname{codim}(S_1 \cap \cdots \cap S_m) \le \sum_{i=1}^{m} \operatorname{codim} S_i = m(n-k).$$

Condition (3.1) therefore implies

$$n \le m(n-k),$$

so

$$m \ge \left\lceil \frac{n}{n-k} \right\rceil. \tag{3.2}$$

Notice that this lower bound already holds when the competing sets are restricted to ellipsoids, since the competitor $E+z$ is itself an ellipsoid.

For the reverse inequality, first consider the unit ball

$$B^n = \{x \in \mathbb{R}^n : |x| < 1\}.$$

Put

$$m = \left\lceil \frac{n}{n-k} \right\rceil.$$

Partition the coordinate indices into m disjoint sets

$$\{1, \dots, n\} = I_1 \,\dot\cup \cdots \dot\cup\, I_m$$

with $|I_i| \le n-k$. For each i, choose an $(n-k)$-dimensional subspace V_i containing

$$\operatorname{span}\{e_j : j \in I_i\},$$

and put $S_i = V_i^{\perp}$. Then $\dim S_i = k$.

Define

$$f_1(x) = 1 - \sum_{j \in I_1} x_j^2, \qquad f_i(x) = -\sum_{j \in I_i} x_j^2 \quad (i \ge 2).$$

Each f_i depends only on the projection of x onto $V_i = S_i^{\perp}$, and hence is constant on affine subspaces parallel to S_i. Moreover,

$$\sum_{i=1}^{m} f_i(x) = 1 - |x|^2.$$

Thus

$$B^n = \left\{ x : \sum_{i=1}^{m} f_i(x) > 0 \right\},$$

so B^n is additive with respect to $S_1, \dots, S_m$. By Theorem 2.1, these m X-rays determine the ball among measurable sets.

Finally, every ellipsoid is a nonsingular affine image of B^n. Additivity is preserved by nonsingular affine maps: if a function is constant on affine subspaces parallel to S_i, then after an affine change of variables it is constant on affine subspaces parallel to the image of S_i. Hence the same number of X-rays verifies every ellipsoid. Together with (3.2), this proves the theorem both among measurable sets and among ellipsoids. □

Remark 3.2. For ordinary X-rays, $k = 1$, Theorem 3.1 gives two X-rays for every $n \geq 2$. The two directions may depend on the ellipsoid; this is why the result does not conflict with the three-direction determination theorem of Gardner [5].

4 The X-ray of an ellipsoid

The determination problem is governed by quadratic forms. We first record a standard section formula.

Let C be a positive-definite symmetric $n \times n$ matrix, and set

$$E_C = \{x \in \mathbb{R}^n : x^T C^{-1} x \leq 1\}. \tag{4.1}$$

Let S be a k-dimensional subspace and $V = S^{\perp}$. Denote by

$$C_V = P_V C|_V$$

the compression of C to V.

Lemma 4.1. *For $y \in V$,*

$$X_S E_C(y) = \kappa_k \left(\frac{\det C}{\det C_V} \right)^{1/2} \left(1 - y^T C_V^{-1} y\right)_+^{k/2}, \tag{4.2}$$

where κ_k is the volume of the k-dimensional unit ball and $a_+ = \max\{a, 0\}$.

Proof. Choose orthonormal coordinates adapted to $\mathbb{R}^n = V \oplus S$ and write

$$C^{-1} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}.$$

A point of $y + S$ has coordinates (y, z), and it belongs to E_C exactly when

$$y^T A y + 2y^T B z + z^T D z \leq 1.$$

Completing the square in z gives

$$(z + D^{-1}B^T y)^T D(z + D^{-1}B^T y) \le 1 - y^T(A - BD^{-1}B^T)y. \tag{4.3}$$

The block inverse formula, or equivalently the Schur-complement identity, gives

$$A - BD^{-1}B^T = C_V^{-1}. \tag{4.4}$$

Hence the section is a k-dimensional ellipsoid whose volume is

$$\kappa_k(\det D)^{-1/2}\left(1 - y^T C_V^{-1} y\right)_+^{k/2}.$$

A second Schur-complement identity gives

$$\det D = \frac{\det C_V}{\det C},$$

and (4.2) follows. For the block-matrix identities used here, see Horn and Johnson [7, Sections 0.7 and 7.7]. □

Thus an S-X-ray of a centered ellipsoid is determined by the pair

$$C|_{S^\perp}, \qquad \det C. \tag{4.5}$$

For a translated ellipsoid, the X-ray also records the projection of the center onto $S^\perp$.

5 A linear-algebraic characterization of determination

Fix k-dimensional subspaces $S_1, \dots, S_m$, and let $V_i = S_i^\perp$. Let Sym_n denote the vector space of real symmetric $n \times n$ matrices. Consider the restriction map

$$R : \mathrm{Sym}_n \longrightarrow \bigoplus_{i=1}^{m} \mathrm{Sym}(V_i), \qquad R(C) = \left(C|_{V_1}, \dots, C|_{V_m}\right). \tag{5.1}$$

Theorem 5.1. *The following are equivalent.*

1. *The X-rays parallel to $S_1, \dots, S_m$ determine every ellipsoid in $\mathbb{R}^n$ among measurable sets.*
2. *They determine every ellipsoid among ellipsoids.*
3. *The restriction map R in (5.1) is injective.*
4. *There is no nonzero real quadratic form that vanishes identically on every $V_i = S_i^\perp$.*

Proof. The implication (1)⇒(2) is immediate.

Assume (2). We first claim that

$$V_1 + \cdots + V_m = \mathbb{R}^n. \tag{5.2}$$

Otherwise there is a nonzero vector

$$z \in (V_1 + \cdots + V_m)^\perp = S_1 \cap \cdots \cap S_m,$$

and an ellipsoid and its translate by z have all the same prescribed X-rays.

Suppose now that R is not injective, and choose $0 \neq D \in \ker R$. Thus

$$D|_{V_i} = 0 \qquad (i = 1, \dots, m). \tag{5.3}$$

The symmetric matrix D must be indefinite. Indeed, if D were positive semidefinite, then $x^T D x = 0$ would imply $Dx = 0$. Equation (5.3) would therefore give $V_i \subset \ker D$ for every i, and (5.2) would force $D = 0$. The same argument excludes negative semidefiniteness.

Consider

$$C_t = I + tD.$$

Because D has both positive and negative eigenvalues, the set of t for which C_t is positive definite is a bounded open interval (a, b) containing 0. Moreover,

$$\det C_t \longrightarrow 0 \quad \text{as } t \to a^+ \text{ or } t \to b^-.$$

Choose $t_0 \in (a, b)$ and then choose c with

$$0 < c < \det C_{t_0}.$$

By the intermediate value theorem, the equation $\det C_t = c$ has one solution $t_1 \in (a, t_0)$ and another $t_2 \in (t_0, b)$. Hence $t_1 \neq t_2$ and

$$\det C_{t_1} = \det C_{t_2}. \tag{5.4}$$

Also, by (5.3),

$$C_{t_1}|_{V_i} = C_{t_2}|_{V_i} \qquad (i = 1, \dots, m). \tag{5.5}$$

Lemma 4.1, together with (5.4)–(5.5), shows that the two distinct centered ellipsoids $E_{C_{t_1}}$ and $E_{C_{t_2}}$ have identical prescribed X-rays. This contradicts (2). Hence R is injective, proving (2)⇒(3).

The equivalence of (3) and (4) is immediate: $D \in \ker R$ exactly when the quadratic form $x \mapsto x^T D x$ vanishes on every V_i.

It remains to prove (3)⇒(1). Equip Sym_n with the trace inner product $\langle A, B \rangle = \mathrm{tr}(AB)$. The adjoint of R has image equal to the sum of the spaces of quadratic forms depending only on the variables in V_i. Since R is injective, its adjoint is surjective. Thus every quadratic form q on $\mathbb{R}^n$ can be written

$$q(x) = q_1(P_{V_1} x) + \cdots + q_m(P_{V_m} x) \tag{5.6}$$

for suitable quadratic forms q_i on V_i.

Injectivity of R also implies (5.2): if $V_1 + \cdots + V_m$ were proper, any nonzero linear functional annihilating this sum would yield a nonzero rank-one quadratic form vanishing on every V_i. Hence every linear functional on $\mathbb{R}^n$ is a sum of linear functionals that depend respectively only on $V_1, \dots, V_m$. A constant term may be placed in any one summand. Consequently every affine quadratic polynomial Q admits a decomposition

$$Q(x) = f_1(P_{V_1} x) + \cdots + f_m(P_{V_m} x), \tag{5.7}$$

where each f_i is an affine quadratic polynomial on V_i.

Every ellipsoid is the positivity set of an affine quadratic polynomial with negative-definite quadratic part. Equation (5.7) therefore represents every ellipsoid as an additive set with respect to $S_1, \dots, S_m$. Theorem 2.1 gives uniqueness among measurable sets. Thus (3)⇒(1). □

Remark 5.2. The lower-bound part of Theorem 5.1 is stronger than required for Problem 2.12. If R is not injective, the prescribed X-rays fail to determine ellipsoids even among ellipsoids. If R is injective, additive-set uniqueness upgrades the conclusion to uniqueness among all measurable sets.

6 Reduction to quadrics containing linear spaces

Condition (4) in Theorem 5.1 has a projective interpretation. A nonzero homogeneous quadratic form on $\mathbb{R}^n$ vanishing on V_i defines a projective quadric in $\mathbb{P}^{n-1}$ containing

$$\mathbb{P}(V_i) \cong \mathbb{P}^{n-k-1}.$$

Thus the determination problem asks for the least number of projective $(n-k-1)$-planes in $\mathbb{P}^{n-1}$ whose union is contained in no quadric.

Carlini, Catalisano, and Geramita [1] determine the degree-two part of the ideal of an arbitrary generic configuration of projective linear spaces; see in particular their Theorem 4.3. We shall apply their theorem only in the equal-dimensional case.

There are two elementary points about genericity that are worth recording.

First, for a fixed number of subspaces, the restriction map R depends polynomially on local coordinates in the relevant product of Grassmannians. Its rank is therefore maximal on a nonempty Zariski-open set. Equivalently, the kernel dimension is minimal for a generic configuration. Consequently, if a generic configuration still has a nonzero quadratic form vanishing on all its members, then no exceptional configuration can have an injective restriction map.

Second, although [1] works over an algebraically closed field, their existence results yield real configurations as well. Indeed, full rank of R is the nonvanishing of at least one maximal minor, which is a polynomial with real coefficients in real local Grassmannian coordinates. If this polynomial is not identically zero over $\mathbb{C}$, it cannot vanish at every real point of an open coordinate chart. Hence some real configuration has full rank.

7 Determination

Theorem 7.1 (Determination number)**.** *Let $1 \leq k \leq n-1$. The minimum number $N(n,k)$ of fixed k-dimensional subspaces whose X-rays determine every ellipsoid in $\mathbb{R}^n$ among measurable sets satisfies*

$$\boxed{N(2,1) = 3,} \tag{7.1}$$

and, for $n \geq 3$,

$$\boxed{N(n,k) = \begin{cases} 3, & k \leq (n+1)/3, \\ 4, & (n+1)/3 < k < n/2, \\ \left\lceil \dfrac{n(n+1)}{(n-k)(n-k+1)} \right\rceil, & k \geq n/2. \end{cases}} \tag{7.2}$$

The same number is required if the competing sets are restricted to ellipsoids.

For $n \geq 3$ the two thresholds in (7.2) satisfy $(n+1)/3 < n/2$. They have a simple geometric meaning. The threshold $(n+1)/3$ is where three complementary subspaces cease to suffice, while $n/2$ is the point at which two generic complementary projective subspaces become disjoint. The proof below naturally uses the latter distinction and then translates back to the more readable ranges in (7.2).

Proof. By Theorem 5.1, $N(n,k)$ is the least number of $(n-k)$-dimensional vector subspaces of $\mathbb{R}^n$ on whose union no nonzero quadratic form vanishes identically. After projectivization, these become projective $(n-k-1)$-planes in $\mathbb{P}^{n-1}$. The proof is most naturally divided according to whether $2k \geq n$ or $2k < n$, because this records whether two generic complementary projective subspaces are disjoint. At the end we translate this geometric split into the simpler ranges displayed in (7.2).

Case 1: $2k \geq n$**.**

Two generic projective $(n-k-1)$-planes in $\mathbb{P}^{n-1}$ are disjoint, because

$$2(n-k-1) < n-1$$

is equivalent to $2k \geq n$. The vector space of quadratic forms on $\mathbb{R}^n$ has dimension

$$\binom{n+1}{2},$$

whereas restriction to one $(n-k)$-dimensional vector subspace gives at most

$$\binom{n-k+1}{2}$$

independent conditions. Thus necessarily

$$m\binom{n-k+1}{2} \geq \binom{n+1}{2},$$

which gives

$$m \geq \left\lceil \frac{n(n+1)}{(n-k)(n-k+1)} \right\rceil. \tag{7.3}$$

The first clause of Carlini–Catalisano–Geramita, Theorem 4.3, states that for a generic disjoint configuration the expected conditions on quadrics are independent: in their notation,

$$\dim(I_\Lambda)_2 = \max\left\{\binom{N+2}{2} - \sum_i \binom{m_i+2}{2}, 0\right\}$$

when the two largest component dimensions satisfy $m_1 + m_2 < N$ [1, Theorem 4.3]. Substituting ambient projective dimension $N = n-1$ and equal component dimension $m_i = n-k-1$ shows that equality in (7.3) is attainable. This proves the third line of (7.2), namely the range $k \geq n/2$.

Case 2: $2k < n$**.**

Two subspaces never suffice. Indeed, if V_1, V_2 are proper vector subspaces, choose nonzero linear forms ℓ_i vanishing on V_i. Then the nonzero quadratic form $\ell_1\ell_2$ vanishes on $V_1 \cup V_2$. Hence

$$N(n,k) \geq 3. \tag{7.4}$$

We next test three generic subspaces. In the notation of Carlini–Catalisano–Geramita, the ambient projective dimension is

$$N = n-1,$$

each component dimension is

$$m = n-k-1,$$

and $s = 3$. Since $2k < n$, the components intersect for dimension reasons. Their parameters in Theorem 4.3 are

$$\tau = 3, \qquad v = 2(n-2k). \tag{7.5}$$

If $3k \leq n$, then

$$v \geq m + 1,$$

so Theorem 4.3(i) of [1] gives

$$\dim(I_\Lambda)_2 = 0.$$

Thus three generic subspaces suffice.

It remains to consider $3k \geq n + 1$. In this range, the parameters fall under Theorem 4.3(ii)(c), with $\tau = s = 3$. Its formula becomes

$$\dim(I_\Lambda)_2 = \max\left\{\binom{n+1}{2} - 3\binom{n-k+1}{2} + 3\binom{n-2k+1}{2}, 0\right\}. \tag{7.6}$$

A direct expansion gives

$$\binom{n+1}{2} - 3\binom{n-k+1}{2} + 3\binom{n-2k+1}{2} = \frac{(3k-n)(3k-n-1)}{2}. \tag{7.7}$$

Therefore three generic subspaces lie on no nonzero quadric exactly when

$$3k = n + 1$$

in this range. Combining this with the previous subcase $3k \leq n$, we conclude that

$$N(n,k) = 3 \quad \text{when } 2k < n \text{ and } 3k \leq n + 1. \tag{7.8}$$

If $3k > n + 1$, the quantity in (7.7) is positive. Since generic rank is maximal, a positive generic kernel implies that no triple, generic or exceptional, can make the restriction map injective. Hence in this remaining range

$$N(n,k) \geq 4. \tag{7.9}$$

We finally show that four generic subspaces always suffice when $2k < n$. Now $s = 4$, $\tau = 4$, and

$$v = 3(n - 2k). \tag{7.10}$$

If

$$5k \leq 2n,$$

then

$$v \geq m + 1,$$

so Theorem 4.3(i) of [1] gives $\dim(I_\Lambda)_2 = 0$.

Suppose instead that

$$5k > 2n.$$

Because n and k are integers,

$$5k \geq 2n + 1. \tag{7.11}$$

The inequality $2k < n$ implies $2k \leq n - 1$, and hence

$$n - 2k + 1 \leq 3(n - 2k) = v. \tag{7.12}$$

On the other hand, (7.11) is equivalent to

$$v \leq n - k - 1 = m. \tag{7.13}$$

Thus

$$2m - N + 2 \le v \le m,$$

so Theorem 4.3(ii) applies. Since $\tau = 4$, part (ii)(a) gives again

$$\dim(I_\Lambda)_2 = 0.$$

Hence four generic subspaces suffice throughout the range $2k < n$. In this range, the condition $3k \le n+1$ is simply $k \le (n+1)/3$, while $3k > n+1$ is $k > (n+1)/3$; and $2k < n$ is $k < n/2$. Thus (7.8)–(7.9) give exactly the first two ranges in (7.2). The case $n = 2$, $k = 1$ belongs to Case 1 and gives $N(2,1) = 3$. Hence (7.1)–(7.2) follow.

The final assertion, concerning determination among ellipsoids only, follows from Theorem 5.1: the same injectivity condition is necessary already for uniqueness within the class of ellipsoids. □

8 Successive determination

We now consider the intermediate, adaptive notion introduced by Edelsbrunner and Skiena and used in the X-ray setting by Gardner and Gritzmann [2, 3]. A *successive strategy* starts with an arbitrarily prescribed k-dimensional subspace S_1. After the X-ray parallel to S_1 has been observed, a second subspace S_2 may be chosen using only that X-ray. In general, S_j may depend only on the first $j-1$ X-rays. The strategy succeeds if, after finitely many steps, the unknown ellipsoid is determined among measurable sets.

Theorem 8.1 (Successive determination number)**.** *Let $1 \le k \le n-1$. The minimum number $S(n,k)$ of k-dimensional X-rays needed to successively determine every ellipsoid in $\mathbb{R}^n$ is exactly the determination number $N(n,k)$ in Theorem 7.1. Thus $S(2,1) = 3$, and for $n \ge 3$,*

$$\boxed{S(n,k) = \begin{cases} 3, & k \le (n+1)/3, \\ 4, & (n+1)/3 < k < n/2, \\ \left\lceil \dfrac{n(n+1)}{(n-k)(n-k+1)} \right\rceil, & k \ge n/2. \end{cases}} \tag{8.1}$$

The same number is required whether the competitors are arbitrary measurable sets or ellipsoids. In fact, the lower bound below already holds when competitors are restricted to centered ellipsoids.

Proof. We first prove the lower bound. We use the following slightly strengthened induction statement: if $m < N(n,k)$, then for every successive strategy using m X-rays there are two distinct positive-definite matrices C_0, C_1 with the same determinant such that the centered ellipsoids E_{C_0} and E_{C_1} give exactly the same first m X-rays under the strategy.

For $m = 0$ there are no X-rays to compare, and we may simply take

$$C_0 = I, \qquad C_1 = \operatorname{diag}(2, 1/2, 1, \ldots, 1),$$

which are distinct, positive definite, and have the same determinant. Suppose now that $1 \le m < N(n,k)$ and assume the strengthened assertion has been proved for $m-1$. Consider any successive strategy of length m. By the induction hypothesis, there are two distinct positive-definite matrices C_0 and C_1, with

$$\det C_0 = \det C_1 =: c,$$

such that the centered ellipsoids E_{C_0} and E_{C_1} give exactly the same first $m-1$ X-rays under this strategy. Consequently the strategy chooses the same first $m-1$ subspaces for both ellipsoids; write $V_i = S_i^{\perp}$ for their orthogonal complements.

Equality of these X-rays and Lemma 4.1 imply

$$C_0|_{V_i} = C_1|_{V_i} \quad (i = 1, \dots, m-1), \qquad \det C_0 = \det C_1 = c. \tag{8.2}$$

Indeed, the positivity set (equivalently, the essential support) of the continuous X-ray function determines $C|_{V_i}$; once this restriction is known, its multiplicative factor in (4.2) determines $\det C$.

Set

$$C_* = \frac{1}{2}(C_0 + C_1).$$

The function $C \mapsto \log\det C$ is strictly concave on the cone of positive-definite matrices. For completeness, along the segment $C_t = (1-t)C_0 + tC_1$ one has

$$\frac{d^2}{dt^2}\log\det C_t = -\operatorname{tr}\Big(\big(C_t^{-1/2}(C_1 - C_0)C_t^{-1/2}\big)^2\Big) < 0$$

when $C_0 \neq C_1$. Hence

$$\det C_* > c. \tag{8.3}$$

By (8.2), C_* has the same restrictions as C_0 and C_1 on $V_1, \dots, V_{m-1}$.

Because the first $m-1$ X-rays are identical, the strategy chooses the same final subspace S_m for every ellipsoid having this common transcript. Put $V_m = S_m^{\perp}$. Since $m < N(n,k)$, Theorem 7.1 and Theorem 5.1 imply that the restriction map

$$D \longmapsto (D|_{V_1}, \dots, D|_{V_m})$$

has a nonzero kernel. This kernel contains an indefinite matrix. To see this, choose a nonzero symmetric matrix D in the kernel. If D is already indefinite, there is nothing to prove. If, say, D is positive semidefinite, then every V_i is contained in $\ker D$, so $V_1 + \cdots + V_m$ is a proper subspace. Choose $0 \neq z$ perpendicular to this sum and choose $w \perp z$, $w \neq 0$. The symmetric matrix corresponding to the quadratic form

$$2(z \cdot x)(w \cdot x)$$

also vanishes on every V_i and is indefinite. The negative-semidefinite case is identical. We may therefore fix an indefinite $0 \neq D$ satisfying

$$D|_{V_i} = 0 \qquad (i = 1, \dots, m). \tag{8.4}$$

Consider the line $C_* + tD$. Since D is indefinite, the set of t for which $C_* + tD$ is positive definite is a bounded interval (a,b) containing 0, and

$$\det(C_* + tD) \longrightarrow 0$$

at both endpoints. In view of (8.3), the intermediate value theorem gives numbers

$$a < t_- < 0 < t_+ < b$$

such that

$$\det(C_* + t_-D) = \det(C_* + t_+D) = c. \tag{8.5}$$

Put

$$C_- = C_* + t_- D, \qquad C_+ = C_* + t_+ D.$$

By (8.4), C_- and C_+ have the same restrictions on every V_i, and by (8.5) they have the same determinant. Lemma 4.1 therefore shows that E_{C_-} and E_{C_+} have identical first m X-rays. Moreover, for $i < m$ these X-rays are exactly the common transcript in (8.2), so the strategy really does choose the same final subspace S_m for both. Since $C_- \neq C_+$, the strategy fails. This completes the induction and proves that at least $N(n,k)$ successive X-rays are necessary.

For the upper bound, take a determining family of $N(n,k)$ complementary subspaces supplied by Theorem 7.1; denote them by

$$V_1^0, \dots, V_{N(n,k)}^0.$$

Let the first k-dimensional X-ray subspace S_1 be prescribed arbitrarily, and put $V_1 = S_1^\perp$. Choose an invertible linear map T with

$$TV_1^0 = V_1.$$

Then no nonzero quadratic form vanishes on all the subspaces

$$TV_1^0, \dots, TV_{N(n,k)}^0 :$$

otherwise composition with T would give a nonzero quadratic form vanishing on all the original V_i^0. Hence, by Theorem 5.1, the k-dimensional subspaces

$$S_i = (TV_i^0)^\perp$$

form a determining family and include the prescribed first subspace S_1. Thus, after any first X-ray, one may simply use these remaining fixed subspaces; no adaptation is actually needed. Therefore $N(n,k)$ successive X-rays suffice. □

Remark 8.2. The proof isolates the reason successive determination does not improve the number for ellipsoids. If two positive-definite matrices survive the X-rays already taken, their midpoint has strictly larger determinant. Any further family of restrictions with a nontrivial quadratic kernel contains an indefinite invisible direction, and moving from the midpoint in the two opposite directions along that kernel produces two new positive-definite matrices with the original determinant. Thus an adaptive last X-ray cannot remove an ambiguity unless the final restriction map is already injective.

9 Examples and comparison of the three problems

For ordinary X-rays, $k = 1$, Theorem 7.1 gives

$$N(n,1) = 3 \qquad (n \geq 2),$$

in agreement with Gardner's theorem [5]. By contrast, Theorem 3.1 gives two X-rays for verification of a known ellipsoid.

At the other extreme, for hyperplane X-rays $k = n-1$,

$$N(n, n-1) = \binom{n+1}{2},$$

whereas the verification number is only n.

Some small values of the determination number are displayed below.

$n\backslash k$	1	2	3	4	5	6	7
2	3						
3	3	6					
4	3	4	10				
5	3	3	5	15			
6	3	3	4	7	21		
7	3	3	4	5	10	28	
8	3	3	3	4	6	12	36

Only entries with $1 \le k \le n-1$ are shown.

The case $(n,k) = (7,3)$ illustrates why a simple dimension count is insufficient. Three restrictions would appear to provide

$$3\binom{5}{2} = 30$$

scalar conditions on a 28-dimensional space of quadratic forms, but Theorem 7.1 gives

$$N(7,3) = 4.$$

The reason is that the complementary 4-dimensional vector subspaces necessarily intersect, and the resulting conditions on quadratic forms are not independent. This is precisely the phenomenon measured by the Hilbert-function theorem of Carlini–Catalisano–Geramita.

For convenient comparison, the three answers are

$$\boxed{V(n,k) = \left\lceil \frac{n}{n-k} \right\rceil}$$

for verification of an individual ellipsoid, and

$$\boxed{N(n,k) = \begin{cases} 3, & k \le (n+1)/3, \\ 4, & (n+1)/3 < k < n/2, \\ \left\lceil \dfrac{n(n+1)}{(n-k)(n-k+1)} \right\rceil, & k \ge n/2 \end{cases} \qquad (n \ge 3)}$$

for determination by one fixed family of X-ray subspaces. For $n = 2$, $N(2,1) = 3$. By Theorem 8.1, the successive determination number is the same:

$$S(n,k) = N(n,k).$$

All three numerical answers are unchanged if "among measurable sets" is replaced by "among ellipsoids." For verification this is already forced by the translation lower bound in Theorem 3.1; for determination and successive determination it follows from Theorems 7.1 and 8.1. Thus verification can require fewer X-rays, but successive choice of the subspaces gives no numerical improvement over determination for ellipsoids.

10 Concluding remark

The tomography ingredient in the proof—additive-set uniqueness—was already available before the second edition of *Geometric Tomography*; see [4, 5]. The interactive notion of successive determination had also been introduced earlier and developed for polytopes; see [2, 3]. The ingredient that appears to complete the general determination problem for ellipsoids is the later computation by Carlini, Catalisano, and Geramita of the quadrics through generic configurations of linear spaces [1]. Theorem 5.1 explains why that algebraic-geometric result is exactly the required one, while Theorem 8.1 shows that strict concavity of the determinant prevents adaptive choice of the X-ray subspaces from lowering the number.

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