A Counterexample for an Infinite Set of Divergent-Beam Sources

Abstract

For a source point $p \in \mathbb{R}^n$, let

$$D_p f(u) = \int_0^\infty f(p + tu)\, dt, \qquad u \in \mathbb{S}^{n-1},$$

be the divergent-beam transform. Problem C.1 in *Geometric Tomography* asks whether a bounded measurable function of bounded support must vanish almost everywhere if $D_p f = 0$ for every p in an arbitrary infinite set of source points. We give a negative answer in every dimension $n \geq 2$. In fact, for

$$p_j = (2^{-j}, 0, \ldots, 0), \qquad j \geq 1,$$

there is a nonzero $f \in C_c^\infty(\mathbb{R}^n)$ such that $D_{p_j} f = 0$ for all j.

1 Background and notation

We use only a small amount of Fourier analysis and distribution theory, but we recall the terminology because it may be less familiar in convex geometry. The Schwartz space $\mathcal{S}(\mathbb{R}^m)$ consists of the C^∞ functions g for which $x^\mu \partial^\nu g(x)$ is bounded for all multi-indices μ, ν; its continuous dual $\mathcal{S}'(\mathbb{R}^m)$ is the space of tempered distributions. The Fourier transform is an automorphism of both spaces. Standard references are Hörmander [3, Chapter 7] and Reed–Simon [4, Chapter IX].

A smooth function q on frequency space will be called a *Schwartz multiplier* if q and all its derivatives have at most polynomial growth. Equivalently for the present purpose, multiplication by q maps $\mathcal{S}$ continuously into itself: the Leibniz rule shows that the rapid decay of a Schwartz function dominates the polynomial growth of every derivative of q. Thus, if q is such a multiplier and $\widehat{\phi} \in \mathcal{S}$, then $\widehat{\phi}\, q \in \mathcal{S}$. The inverse Fourier transform of q itself need not be an ordinary function; it is, however, a tempered distribution. We shall construct such a multiplier with prescribed zero hyperplanes and then regularize its inverse Fourier transform by convolution with a compactly supported C^∞ function.

For an open cone $\Gamma \subset \mathbb{R}^m$, the set $\mathbb{R}^m + i\Gamma \subset \mathbb{C}^m$ is called a *tube domain*. A basic Fourier–Laplace principle says that holomorphic extension of a Fourier transform to a tube reflects one-sided support of the inverse transform. This is a standard Paley–Wiener type phenomenon; see, for example, Reed–Simon [4, Section IX.5] or Vladimirov [5]. To avoid importing a theorem whose sign convention may differ from ours, we prove below exactly the support statement that we need by an elementary contour-shifting argument (Lemma 2).

The divergent-beam transform and its relation to the Radon transform are developed systematically by Hamaker, Smith, Solmon and Wagner [2]. Their uniqueness results also explain why the geometry of the source set relative to an enclosing body is important; our example lies in the complementary boundary/accumulation regime.

2 Projective reduction

Write points of $\mathbb{R}^n$ as (x', y), where $x' \in \mathbb{R}^{n-1}$. For $a \in \mathbb{R}$, set

$$p_a = (ae_1, 0).$$

For $s \in \mathbb{R}^{n-1}$, the upward ray through p_a with slope s is

$$x' = ae_1 + sy, \qquad y > 0.$$

Introduce projective coordinates

$$X = \frac{x'}{y}, \qquad Y = \frac{1}{y},$$

and, for a function f supported in $\{y \geq 0\}$, define

$$F(X, Y) = Y^{-2} f(X/Y, 1/Y), \qquad Y > 0.$$

Then

$$\int_0^\infty f(ae_1 + sy, y)\, dy = \int_0^\infty F(s + ae_1 Y, Y)\, dY. \tag{1}$$

Indeed, this follows from $Y = 1/y$ and $dy = -Y^{-2} dY$.

Define

$$P_a F(s) = \int_0^\infty F(s + ae_1 Y, Y)\, dY.$$

We use the Fourier transform

$$\widehat{F}(\xi, \eta) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} F(X, Y) e^{-i(X \cdot \xi + Y\eta)}\, dY\, dX.$$

A change of variables $X = s + ae_1 Y$ gives

$$\widehat{P_a F}(\xi) = \widehat{F}(\xi, -a\xi_1). \tag{2}$$

Thus it is enough to construct a nonzero Schwartz function F satisfying

$$\widehat{F}(\xi, -2^{-j}\xi_1) = 0 \qquad (j \geq 1) \tag{3}$$

and

$$\operatorname{supp} F \subset \{(X, Y) : Y \geq c,\ |X| \leq MY\} \tag{4}$$

for some $c, M > 0$.

3 A two-dimensional holomorphic multiplier

Put $a_j = 2^{-j}$. Fix

$$0 < \beta < \alpha < 1, \qquad \delta = \alpha - \beta,$$

for example $\alpha = 2/3$ and $\beta = 1/3$. The largest open cone of imaginary directions needed below is

$$\Gamma = \{(v_1, v_2) \in \mathbb{R}^2 : v_2 < 0,\ v_2 + a_1 v_1 < 0\}, \qquad a_1 = \tfrac{1}{2}. \tag{5}$$

If $z = (z_1, z_2) \in \mathbb{R}^2 + i\Gamma$, then

$$\Im(z_2 + a_j z_1) < 0 \qquad (j \geq 1).$$

Indeed, for $\Im z_1 \leq 0$ this follows from $\Im z_2 < 0$, whereas for $\Im z_1 > 0$ the strongest inequality is the one with the largest a_j, namely $a_1 = 1/2$. Consequently

$$i(z_2 + a_j z_1)$$

lies in the open right half-plane for every j. Using the principal branch there, define

$$S(z) = \sum_{j=1}^{\infty} a_j^{\beta} \{i(z_2 + a_j z_1)\}^{-\alpha}, \qquad Q(z) = e^{-S(z)}. \tag{6}$$

The series converges normally (that is, uniformly and absolutely) on compact subsets of the tube. Hence it defines a holomorphic function there, and $\operatorname{Re} S(z) > 0$, so

$$|Q(z)| \leq 1. \tag{7}$$

For real (ξ, η) away from the lines $\eta = -a_j \xi$, let $q(\xi, \eta)$ be the boundary value of Q. Since for real $t \neq 0$,

$$\operatorname{Re}(it)^{-\alpha} = \cos(\pi\alpha/2)|t|^{-\alpha},$$

we have

$$-\log|q(\xi, \eta)| = c_\alpha \sum_{j=1}^{\infty} a_j^{\beta} |\eta + a_j \xi|^{-\alpha}, \qquad c_\alpha = \cos(\pi\alpha/2) > 0. \tag{8}$$

We extend q by zero on every line $\eta = -a_j \xi$, on the limiting line $\eta = 0$, and at the origin.

Lemma 1 (Multiplier lemma)**.** *The extended function q belongs to $C^\infty(\mathbb{R}^2)$. It is flat on every line $\eta = -a_j \xi$ and on $\eta = 0$. Moreover, for every multi-index γ, there are constants C_γ, N_γ such that*

$$|\partial^\gamma q(\xi, \eta)| \leq C_\gamma (1 + |\xi| + |\eta|)^{N_\gamma}.$$

Proof. Only the accumulation at $\eta = 0$ requires a uniform argument. Work first in the half-plane $\xi > 0$. Put

$$s = -\eta/\xi, \qquad \lambda = \xi^{-\alpha}.$$

Then

$$q(\xi, \eta) = e^{-\lambda A_+(s)}, \qquad A_+(s) = \sum_{j=1}^{\infty} a_j^{\beta} \{i(a_j - s)\}^{-\alpha}. \tag{9}$$

In the half-plane $\xi < 0$, put again $s = -\eta/\xi$ and $\lambda = |\xi|^{-\alpha}$. Since $\eta + a_j \xi = |\xi|(s - a_j)$ there,

$$q(\xi, \eta) = e^{-\lambda A_-(s)}, \qquad A_-(s) = \sum_{j=1}^{\infty} a_j^{\beta} \{i(s - a_j)\}^{-\alpha}. \tag{10}$$

For either sign, write $A = A_+$ or A_- as appropriate. In both cases

$$\operatorname{Re} A(s) = c_\alpha \sum_{j=1}^{\infty} a_j^{\beta} |s - a_j|^{-\alpha}, \tag{11}$$

and, for each $m \geq 0$,

$$|A^{(m)}(s)| \leq C_m \sum_{j=1}^{\infty} a_j^{\beta} |s - a_j|^{-\alpha - m}. \tag{12}$$

Assume first that $0 < s < 1/8$, and choose J so that $a_{J+1} < s \leq a_J$. Set

$$N_J = \{j : |j - J| \leq 3\}.$$

Splitting the dyadic series into $j < J - 3$, $j \in N_J$, and $j > J + 3$, elementary geometric-series estimates give

$$|A^{(m)}(s)| \leq C_m s^{-\delta - m} + C_m \sum_{j \in N_J} a_j^{\beta} |s - a_j|^{-\alpha - m}, \tag{13}$$

$$\operatorname{Re} A(s) \geq c s^{-\delta} + c \sum_{j \in N_J} a_j^{\beta} |s - a_j|^{-\alpha}. \tag{14}$$

For $-1/8 < s < 0$, there are no singular points and the simpler estimates

$$|A^{(m)}(s)| \leq C_m |s|^{-\delta - m}, \qquad \operatorname{Re} A(s) \geq c|s|^{-\delta} \tag{15}$$

hold.

We claim that for all integers $r, m \geq 0$,

$$\left| (\lambda \partial_\lambda)^r \partial_s^m e^{-\lambda A(s)} \right| \leq C_{r,m} \lambda^{-m/\delta} \tag{16}$$

whenever $0 < |s| < 1/8$ and $s \neq a_j$. After differentiating, each term is a constant multiple of

$$e^{-\lambda A(s)} \prod_{\nu=1}^{k} \{\lambda A^{(m_\nu)}(s)\}, \tag{17}$$

where $m_\nu \geq 0$ and the sum of the positive s-derivative orders in the product is at most m.

For a regular factor from the first term on the right side of (13),

$$\lambda s^{-\delta - r} = (\lambda s^{-\delta})^{1 + r/\delta} \lambda^{-r/\delta}.$$

A fixed positive fraction of $e^{-c\lambda s^{-\delta}}$ absorbs the power of $\lambda s^{-\delta}$.

For a near-line factor, put $d = |s - a_j|$ and $x = \lambda a_j^{\beta} d^{-\alpha}$. Then

$$\lambda a_j^{\beta} d^{-\alpha - r} = x^{1 + r/\alpha} (\lambda a_j^{\beta})^{-r/\alpha}.$$

A fixed fraction of e^{-cx} absorbs $x^{1+r/\alpha}$. Since $j \in N_J$, we have $a_j \asymp s$. Another fixed fraction of the collective factor $e^{-c\lambda s^{-\delta}}$, together with

$$s^{-p} e^{-c\lambda s^{-\delta}} \leq C_p \lambda^{-p/\delta}, \tag{18}$$

with $p = \beta r/\alpha$, gives

$$\lambda^{-r/\alpha} \lambda^{-\beta r/(\alpha\delta)} = \lambda^{-r/\delta},$$

because

$$\frac{1}{\alpha} + \frac{\beta}{\alpha\delta} = \frac{1}{\delta}.$$

There may be several near-line factors in (17). For fixed r, m their number is bounded independently of J, s, λ. Split the exponential supplied by (14) into sufficiently many fixed positive fractions: one for each singular factor, one for each use of (18), and one final fraction left unused. The preceding estimates then multiply, and the total exponent of $\lambda^{-1/\delta}$ is at most m. This proves (16). Because one fraction of the exponential is left unused, the same expressions tend to zero as $s \to 0$. The identical allocation argument near a fixed $s = a_j$ proves flatness there.

Return now to (ξ, η). On both half-planes $\xi \neq 0$,

$$\partial_\eta = -\xi^{-1}\partial_s, \qquad \partial_\xi = \xi^{-1}\big(-\alpha\lambda\partial_\lambda - s\partial_s\big). \tag{19}$$

Repeated use of (19) and (16) gives polynomial bounds for every mixed derivative in the region $|\eta| < |\xi|/8$, and the unused exponential fraction shows that all derivatives tend to zero on the limiting line $\eta = 0$.

Away from $|\eta| < |\xi|/8$, the accumulating family causes no difficulty: on each compact subset only finitely many source lines occur, while the remaining tail and all its derivatives converge normally. Ordinary exponential flatness handles those finitely many lines and yields polynomial bounds at infinity.

Finally consider $(\xi, \eta) \to (0, 0)$. Off the line $\eta = -a_1\xi$, the first summand in (8) gives

$$\operatorname{Re} S(\xi, \eta) \geq c|\eta + a_1\xi|^{-\alpha} \geq c'(\xi^2 + \eta^2)^{-\alpha/2},$$

because $|\eta + a_1\xi| \leq C(\xi^2 + \eta^2)^{1/2}$. Near any individual source line, its own exponential factor absorbs all singular derivative factors; the displayed fixed-term estimate leaves an additional exponential factor tending to zero faster than every power of $(\xi^2 + \eta^2)^{1/2}$. Hence every derivative tends to zero at the origin. This completes the proof. □

4 A direct cone-support lemma

The following elementary lemma is the support/holomorphy principle used in the construction. We include the proof both for completeness and to make the sign of the polar cone transparent under our Fourier convention $e^{-ix\cdot\xi}$.

Lemma 2 (Tube support). *Let $\Gamma \subset \mathbb{R}^m$ be an open convex cone and let Q be bounded and holomorphic in $\mathbb{R}^m + i\Gamma$, with bounded boundary value q on $\mathbb{R}^m$. Put*

$$C_\Gamma = \{x \in \mathbb{R}^m : x \cdot v \leq 0 \text{ for every } v \in \Gamma\}.$$

If $T = \mathcal{F}^{-1}q$ in the sense of tempered distributions, then

$$\operatorname{supp} T \subset C_\Gamma.$$

Proof. Let $x_0 \notin C_\Gamma$. By the definition of C_Γ, and then by continuity, there are $v \in \Gamma$, a neighborhood U of x_0, and $m_0 > 0$ such that

$$x \cdot v \geq m_0 \qquad (x \in U).$$

Take $\psi \in C_c^\infty(U)$ and, for $t > 0$, set

$$I(t) = \int_{\mathbb{R}^m} Q(\zeta + itv)\widehat{\psi}(-\zeta - itv)\, d\zeta.$$

Since

$$\widehat{\psi}(-\zeta - itv) = \int \psi(x) e^{ix\cdot\zeta} e^{-tx\cdot v}\, dx,$$

repeated integration by parts gives, for every N,

$$|\widehat{\psi}(-\zeta - itv)| \le C_N (1+t)^N e^{-m_0 t} (1+|\zeta|)^{-N}.$$

Thus $I(t)$ is absolutely convergent. If $H(z) = Q(z)\widehat{\psi}(-z)$, the Cauchy–Riemann equations give

$$I'(t) = i \int_{\mathbb{R}^m} v \cdot \nabla_\zeta H(\zeta + itv)\, d\zeta = 0.$$

Differentiation under the integral is justified by Cauchy estimates on a slightly smaller tube and the preceding rapid decay; the last equality is integration of a total derivative. Hence $I(t)$ is constant. The same estimate shows $I(t) \to 0$ as $t \to \infty$, and therefore $I(t) = 0$ for all $t > 0$. Letting $t \downarrow 0$ gives

$$\int_{\mathbb{R}^m} q(\zeta)\widehat{\psi}(-\zeta)\, d\zeta = 0,$$

which, apart from the fixed Fourier normalization constant, says $\langle T, \psi\rangle = 0$. Thus T vanishes in a neighborhood of every point outside C_Γ. $\square$

For the cone Γ in (5), its polar cone in the sense of Lemma 2 is particularly simple:

$$C = \{(X_1, Y) : Y \ge 0,\ 0 \le X_1 \le \tfrac{1}{2} Y\}. \tag{20}$$

Indeed, the vectors $(0, -1)$ force $Y \ge 0$; allowing v_1 to be arbitrarily negative while $v_2 < 0$ forces $X_1 \ge 0$; and approaching the boundary $v_2 = -\frac{1}{2} v_1$ with $v_1 > 0$ gives $X_1 \le Y/2$. Conversely these inequalities imply $X_1 v_1 + Y v_2 \le 0$ for every $v \in \Gamma$.

Applying Lemma 2 to the function Q constructed above therefore gives

$$\operatorname{supp} T_2 \subset C, \qquad T_2 = \mathcal{F}^{-1} q. \tag{21}$$

5 Construction in every dimension

For $n \ge 2$, write the frequency variables as

$$(\xi_1, \xi'', \eta), \qquad \xi'' \in \mathbb{R}^{n-2},$$

and define

$$q_n(\xi_1, \xi'', \eta) = q(\xi_1, \eta).$$

Then, up to the harmless normalization of the Fourier transform,

$$T_n := \mathcal{F}^{-1} q_n = T_2 \otimes \delta_0$$

in the $n-2$ variables dual to ξ''. Consequently

$$\operatorname{supp} T_n \subset \{(X_1, X'', Y) : Y \ge 0,\ 0 \le X_1 \le \tfrac{1}{2} Y,\ X'' = 0\}. \tag{22}$$

Choose $0 \le \phi \in C_c^\infty(\mathbb{R}^n)$, $\phi \not\equiv 0$, supported in a sufficiently small ball centered at $(0, \dots, 0, 1)$ and put

$$F = \phi * T_n.$$

Then

$$\widehat{F} = \widehat{\phi}\, q_n.$$

By Lemma 1, q_n and all its derivatives have polynomial growth; hence $\widehat{F} \in \mathcal{S}(\mathbb{R}^n)$ and therefore $F \in \mathcal{S}(\mathbb{R}^n)$. Also,

$$\operatorname{supp} F \subset \operatorname{supp} \phi + \operatorname{supp} T_n.$$

Thus there are $c, M > 0$ such that

$$Y \geq c, \qquad |X| \leq MY \tag{23}$$

on $\operatorname{supp} F$. The function F is nonzero: otherwise $\widehat{\phi}\, q_n = 0$; since q_n is nonzero on a nonempty open set, the entire function $\widehat{\phi}$ would vanish on a nonempty open subset of $\mathbb{R}^n$, and hence identically, contrary to $\phi \neq 0$.

Moreover,

$$q_n(\xi_1, \xi'', -a_j\xi_1) = 0,$$

so

$$\widehat{F}(\xi, -a_j\xi_1) = 0.$$

By (2),

$$P_{a_j} F = 0 \qquad (j \geq 1). \tag{24}$$

6 The accumulation point belongs to the support

The source points $p_j = 2^{-j} e_1$ converge to the origin. It is useful to record that, for the example obtained below, the origin actually belongs to $\operatorname{supp} f$. Since f will vanish on the half-space $y \leq 0$, the origin is necessarily a boundary point, not an interior point, of the support. We emphasize that this statement does *not* assert that the individual source points p_j belong to $\operatorname{supp} f$.

We prove the corresponding statement before defining f. Suppose, to the contrary, that after reconstruction the origin were omitted from the support. Then for some $r > 0$ the reconstructed function would vanish whenever $|x'|^2 + y^2 < r^2$. In projective coordinates $x' = X/Y$, $y = 1/Y$, this means that every $(X, Y) \in \operatorname{supp} F$ satisfies

$$|X|^2 + 1 \geq r^2 Y^2. \tag{25}$$

The sharp cone estimate (21), together with the compact support of ϕ, implies that on $\operatorname{supp} F$ the coordinates $X_2, \ldots, X_{n-1}$ are bounded and X_1 is bounded below. Hence (25) forces, for all sufficiently large Y,

$$X_1 \geq c_0 Y \tag{26}$$

for some $c_0 > 0$. Indeed, if $|X''| \leq B$ and $X_1 \geq -B$, then (25) gives $X_1^2 \geq r^2 Y^2 - (B^2 + 1)$; for large Y this yields $|X_1| \geq (r/2)Y$, and the lower bound $X_1 \geq -B$ excludes the negative alternative. Geometrically, the unbounded part of $\operatorname{supp} F$ would stay a positive angle away from the positive Y-axis.

We next explain why this support separation gives extra holomorphy. For $z = (z', z_n) \in \mathbb{C}^n$ define, whenever the integral is absolutely convergent,

$$H(z) = \int_{\mathbb{R}^n} F(X, Y) e^{-i(X \cdot z' + Y z_n)}\, dX\, dY. \tag{27}$$

Fix $\tau > 0$. In a sufficiently small neighborhood of

$$z^0 = (-i\tau, 0, \ldots, 0)$$

we have $\Im z_1 < 0$ bounded away from 0 and $|\Im z_n|$ small. On the unbounded part of $\operatorname{supp} F$, (26) then gives

$$X_1 \Im z_1 + Y \Im z_n \le -cY$$

for some $c > 0$, while the remaining imaginary coordinates cause only a bounded exponential factor because $X_2, \ldots, X_{n-1}$ are bounded there. On the bounded part of the support no restriction is needed. Since F is Schwartz, (27) and all of its z-derivatives are therefore dominated by integrable functions locally uniformly in z. Differentiation under the integral shows that H is holomorphic in a neighborhood of z^0.

There is also an overlap with the original tube $\mathbb{R}^2 + i\Gamma$ in the (z_1, z_n) variables. We record explicitly why on that overlap

$$H(z) = \widehat{\phi}(z) Q(z_1, z_n). \tag{28}$$

Let $v = (v_1, v_n) \in \Gamma$, with the remaining imaginary coordinates zero. Since $\operatorname{supp} T_n$ is contained in the polar cone C of Γ, one has $W \cdot v \le 0$ for $W \in \operatorname{supp} T_n$. Hence $Z \cdot v$ is bounded above on $\operatorname{supp} F \subset \operatorname{supp} \phi + \operatorname{supp} T_n$. The Fourier–Laplace transform of $F = \phi * T_n$ is therefore defined at $\zeta + iv$, and the usual convolution identity, first for test functions and then by the definition of convolution with a tempered distribution, gives

$$H(\zeta + iv) = \widehat{\phi}(\zeta + iv)\, Q(\zeta_1 + iv_1, \zeta_n + iv_n).$$

Equivalently, one may obtain (28) by taking boundary values on the real space and applying the identity theorem on the tube. Our choice $\phi \ge 0$, $\phi \not\equiv 0$, ensures

$$\widehat{\phi}(-i\tau, 0, \ldots, 0) = \int \phi(X, Y) e^{-\tau X_1}\, dX\, dY > 0. \tag{29}$$

Thus $\widehat{\phi}$ is nonzero near z^0, and the assumed continuation of H would give a holomorphic continuation of Q through z^0.

We now show that such a continuation is impossible. Restrict to the complex line $z_1 = -i\tau$, with all other frequency variables except z_n equal to zero, and approach $z_n = 0$ from the original tube by putting $z_n = -i\varepsilon$, $\varepsilon > 0$. Then

$$S(-i\tau, -i\varepsilon) = \sum_{j=1}^{\infty} a_j^{\beta} (\varepsilon + a_j \tau)^{-\alpha}. \tag{30}$$

For the tail with $a_j \tau \le \varepsilon$,

$$S(-i\tau, -i\varepsilon) \ge c\, \tau^{-\beta} \varepsilon^{\beta - \alpha}.$$

Consequently, since $\beta < \alpha$,

$$0 < Q(-i\tau, -i\varepsilon) \le \exp\{-c/(\tau^{\beta} \varepsilon^{\alpha - \beta})\}. \tag{31}$$

For completeness, the same dyadic splitting gives, for every $k \ge 0$,

$$\left| \partial_{z_n}^k S(-i\tau, -i\varepsilon) \right| \le C_{k,\tau}\, \varepsilon^{\beta - \alpha - k} \qquad (0 < \varepsilon \le 1). \tag{32}$$

Indeed, differentiation merely changes the exponent $-\alpha$ in (30) to $-\alpha - k$, and splitting the sum at $a_j \asymp \varepsilon/\tau$ gives (32). Faà di Bruno's formula therefore yields, for each m,

$$\left|\partial_{z_n}^m Q(-i\tau, -i\varepsilon)\right| \le C_{m,\tau}\varepsilon^{-N_m} \exp\{-c_\tau \varepsilon^{-(\alpha-\beta)}\} \longrightarrow 0 \qquad (\varepsilon \downarrow 0)$$

for some integer N_m. If $Q(-i\tau, z_n)$ extended holomorphically through $z_n = 0$, every derivative of the continuation would vanish at 0. Its Taylor series would therefore be identically zero near 0, contradicting $Q(-i\tau, -i\varepsilon) > 0$ for $\varepsilon > 0$.

We conclude that the reconstructed function satisfies

$$0 \in \operatorname{supp} f. \tag{33}$$

This observation is not needed for the counterexample itself. Its purpose is to make clear that the accumulation point of the sources lies on the support, although the whole source hyperplane is a zero set of f.

7 Reconstruction and conclusion

Define

$$f(x', y) = \begin{cases} y^{-2} F(x'/y, 1/y), & y > 0, \\ 0, & y \le 0. \end{cases} \tag{34}$$

By (23), f has compact support. Since F is Schwartz, repeated differentiation of (34) yields finite sums of powers of y^{-1} times derivatives of F evaluated at $(x'/y, 1/y)$. Arbitrary-order Schwartz decay shows that every derivative tends to zero as $y \downarrow 0$. Thus

$$f \in C_c^\infty(\mathbb{R}^n),$$

and $f \ne 0$.

For an upward-pointing unit vector $u = (u', u_n)$ with $u_n > 0$, set $s = u'/u_n$. Parametrizing the ray from p_{a_j} by $y = tu_n$ gives

$$D_{p_{a_j}} f(u) = \frac{1}{u_n} \int_0^\infty f(a_j e_1 + sy, y)\, dy = \frac{1}{u_n} P_{a_j} F(s) = 0$$

by (1) and (24). If $u_n < 0$, the ray lies in $y < 0$ apart from its initial point, and if $u_n = 0$ it lies in $y = 0$; in either case the integral is zero. Therefore

$$D_{p_j} f = 0 \qquad (j \ge 1),$$

where $p_j = (2^{-j}, 0, \dots, 0)$.

We have proved the following.

Theorem 1. *For every $n \ge 2$, there exists a nonzero function $f \in C_c^\infty(\mathbb{R}^n)$ and an infinite set*

$$P = \{(2^{-j}, 0, \dots, 0) : j \ge 1\}$$

such that $D_p f = 0$ for every $p \in P$. Moreover, the accumulation point 0 belongs to $\operatorname{supp} f$.

Thus Problem C.1 of *Geometric Tomography* has a negative answer in every dimension $n \ge 2$.

Remark 1. *The sources are $p_j = 2^{-j}e_1$, not $2^{-j}e_n$: they lie in the hyperplane $y = x_n = 0$, whereas e_n is the normal direction used in the projective transformation. The function f vanishes, indeed is flat, on that hyperplane, but $0 \in \operatorname{supp} f$ by (33). Thus nonzero values of f occur arbitrarily close to the accumulation point from the side $y > 0$. We do not claim here that any particular p_j belongs to $\operatorname{supp} f$. Under the projective change of variables the hyperplane $y = 0$ is sent to infinity, and the assertion $0 \in \operatorname{supp} f$ corresponds to unbounded support of F approaching the positive Y-axis. This geometry is consistent with the exterior-source uniqueness results of Hamaker–Smith–Solmon–Wagner [2, Section 5] and with Theorem C.1.2 of [1].*

References

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